Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 5
Due Tuesday, February 21

This week's mastery quiz has three topics. You should definitely submit M2 and S3. If you have a 4/4 on M1—meaning you've gotten it completely right twice—, then you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) $\dfrac{d}{dx}\dfrac{1}{\arcsin(x^2)} =$

Solution:

$$\frac{d}{dx}\frac{1}{\arcsin(x^2)} = \frac{-1}{\arcsin(x^2)^2}\frac{1}{\sqrt{1-x^4}} \cdot 2x.$$

(b) $\displaystyle\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx =$

Solution: We can take $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{-u}{1+u^2}\,du$$

Then we can set $v = 1+u^2$ so that $dv = 2u\,du$ and we get

$$\begin{aligned}\int \frac{-u}{1+u^2}\,du &= \int \frac{-1}{2}\frac{1}{v}\,dv = \frac{-1}{2}\ln|v| + C \\ &= \frac{-1}{2}\ln|1+u^2| + C = \frac{-1}{2}\ln|1+\cos^2(x)| + C.\end{aligned}$$

(c) $\displaystyle\int \frac{x}{9+x^4}\,dx =$

Solution: We can factor a 9 out to get $\frac{1}{9}\frac{x}{1+x^4/9}$. Then we set $u = x^2/3$, and $du = 2x/3\,dx$, and we have

$$\begin{aligned}\int \frac{x}{9+x^4}\,dx &= \int \frac{1}{9}\frac{x}{1+u^2}\frac{3}{2x}\,du \\ &= \int \frac{1}{6}\frac{1}{1+u^2}\,du \\ &= \frac{1}{6}\arctan u + C = \frac{1}{6}\arctan(x^2/3) + C.\end{aligned}$$

M2: Advanced Integration Techniques

(a) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx =$

Solution:

$$\begin{aligned}
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\
x^2+x-4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\
2 &= -2B \Rightarrow B = -1 \\
-4 &= 4C \Rightarrow C = -1 \\
-4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\
\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx \\
&= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C.
\end{aligned}$$

(b) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx =$

Solution: We're going to set $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Then

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int \frac{\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)}\cos(\theta)\,d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\,d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C.
\end{aligned}$$

But we know that $\sin(\theta) = x$, so $x = \arcsin(\theta)$ and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sqrt{1-x^2}}{x}$. Then we get

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = -\frac{\sqrt{1-x^2}}{x} - \arcsin(x) + C.$$

S3: Numeric Integration

(a) Let $f(x) = x^3 + x$. How many intervals do you need with the midpoint rule to approximate $\int_1^2 x^3 + x\,dx$ to within 1/10? Compute the integral with that many integrals. (Feel free to use a calculator to plug values into f, but show every step.)

Solution: We have

$$\begin{aligned} f''(x) &= 6x \\ f'(2) &= 12 \\ |E_M| &\leq \frac{12 \cdot 1^3}{24 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 5 \\ n &> 2 \end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\int_1^2 x^3 + x\,dx \approx \frac{1}{3}f(7/6) + \frac{1}{3}f(9/6) + \frac{1}{3}f(11/6) \approx \frac{1}{3}(2.75 + 4.875 + 8.00) = \frac{1}{3}15.625 \approx 5.21.$$

(Since the true answer is 5.25 this is in fact within our error bound.)

(b) Suppose we have

$$g(0) = 2.4 \qquad g(1) = 4 \qquad g(2) = 2.7 \qquad g(3) = 2.3 \qquad g(4) = 1.7$$

Approximate $\int_0^4 g(x)\,dx$ using the Trapezoid rule, and then using Simpson's rule.

Solution: For the trapezoid rule, we have

$$\begin{aligned} T_4 &= 1 \cdot \frac{2.4 + 4}{2} + 1 \cdot \frac{4 + 2.7}{2} + 1 \cdot \frac{2.7 + 2.3}{2} + 1 \cdot \frac{2.3 + 1.7}{2} \\ &= \frac{1}{2}(6.4 + 6.7 + 5 + 4.0) = \frac{1}{2} \cdot 22.1 = 11.05. \end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}\left(2.4 + 4 \cdot 3 + 2 \cdot 2.7 + 4 \cdot 2.3 + 1.9\right) \\ &= \frac{1}{3}(2.4 + 12 + 5.4 + 9.2 + 1.9) = \frac{1}{3} \cdot 30.9 \approx 10.3. \end{aligned}$$