

Math 1232: Single-Variable Calculus 2  
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Recitation 5

Jay Daigle

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**Problem 1.** We want to find  $\int \frac{x^5 + x - 1}{x^3 + 1} dx$ .

- (a) What's the first tool we need to apply here? (Hint: not partial fractions!)
- (b) Once we get it in a more manageable form, things should simplify out nicely. What is the final integral?

**Solution:**

- (a) Because the numerator is higher degree than the denominator, we need to start with a polynomial long division. We get

$$\frac{x^5 + x - 1}{x^3 + 1} = x^2 - \frac{x^2 - x + 1}{x^3 + 1} = x^2 - \frac{x^2 - x + 1}{(x + 1)(x^2 - x + 1)}.$$

- (b) We don't even need to do a partial fractions decomposition here: instead it just factors. We have

Thus

$$\begin{aligned} \int \frac{x^5 + x - 1}{x^3 + 1} dx &= \int x^2 - \frac{x^2 - x + 1}{(x + 1)(x^2 - x + 1)} dx \\ &= \int x^2 - \frac{1}{x + 1} dx = \frac{x^3}{3} - \ln|x + 1|. \end{aligned}$$

**Problem 2.** We've looked briefly at the integral  $\int \frac{1}{1 + e^x} dx$ . Let's try it again with our new tools.

- (a) Try the substitution  $u = e^x$ . What do you get? What tools can apply to the result?
- (b) Do a partial fractions decomposition to get the integral.

**Solution:**

- (a) If  $u = e^x$  then  $du = e^x dx$ .

$$\int \frac{1}{1 + e^x} dx = \int \frac{1}{1 + e^x} \frac{du}{e^x} = \int \frac{du}{u(1 + u)}.$$

- (b) This looks like a fraction with a factorable denominator. So a partial fractions decomposition gives us

$$\begin{aligned} \frac{1}{u(1 + u)} &= \frac{A}{u} + \frac{B}{1 + u} \\ 1 &= A(1 + u) + B(u) = A + (A + B)u \end{aligned}$$

so  $A = 1$  and  $B = -1$ .

(Alternatively, plugging in  $u = 0$  gives  $1 = A$ , and plugging in  $u = -1$  gives  $1 = -B$ .)

Thus

$$\begin{aligned} \int \frac{1}{1 + e^x} dx &= \int \frac{du}{u(1 + u)} \\ &= \int \frac{1}{u} - \frac{1}{1 + u} du \\ &= \ln |u| - \ln |1 + u| + C = \ln \left| \frac{e^x}{1 + e^x} \right| + C. \end{aligned}$$

**Problem 3** (Bonus). Let's see if we can work out the integral of secant! This isn't at all obvious.

- (a) We want  $\int \sec(x) dx = \int \frac{1}{\cos(x)} dx$ . Since this is a fraction, we can multiply the top and bottom through by  $\cos(x)$ . This makes the expression more complicated, but it does allow us to use a trig identity. What do we get?
- (b) Now we can do a  $u$  substitution. What  $u$  substitution seems reasonable? Does it help us at all?
- (c) Now we can use partial fractions to finish the problem off. We wind up with an awkward answer, but an answer.

- (d) The most common formula for the integral of  $\sec(x)$  is  $\ln|\sec(x) + \tan(x)| + C$ . Is that the same as what you got? (Hint: use logarithm laws and multiplication by the conjugate.)

**Solution:**

- (a) We get  $\int \frac{\cos(x)}{\cos^2(x)} dx$ . The bottom allows us to use the pythagorean identity, and now we're trying to compute  $\int \frac{\cos(x)}{1-\sin^2(x)} dx$ .

- (b) The only really reasonable choice here is  $u = \sin(x)$ , so  $du = \cos(x) dx$ . Then we have

$$\int \frac{\cos(x)}{1-\sin^2(x)} dx = \int \frac{1}{1-u^2} du.$$

- (c) We write

$$\begin{aligned} \frac{1}{1-u^2} &= \frac{A}{1-u} + \frac{B}{1+u} \\ 1 &= A(1+u) + B(1-u). \end{aligned}$$

Plugging in  $u = 1$  gives us that  $2B = 1$  and plugging in  $u = -1$  gives us  $2A = 1$ , so we have

$$\begin{aligned} \int \frac{\cos(x)}{1-\sin^2(x)} dx &= \int \frac{1}{1-u^2} du \\ &= \frac{1}{2} \int \frac{1}{1-u} + \frac{1}{1+u} du \\ &= \frac{1}{2} (-\ln|1-u| + \ln|1+u|) + C \\ &= \frac{1}{2} (-\ln|1-\sin(x)| + \ln|1+\sin(x)|) + C. \end{aligned}$$

- (d) It doesn't look the same, but it is!

$$\begin{aligned} \frac{1}{2} (-\ln|1-\sin(x)| + \ln|1+\sin(x)|) &= \frac{1}{2} \ln \left| \frac{1+\sin(x)}{1-\sin(x)} \right| \\ &= \frac{1}{2} \ln \left| \frac{(1+\sin(x))^2}{1-\sin^2(x)} \right| \\ &= \frac{1}{2} \ln \left| \frac{(1+\sin(x))^2}{\cos^2(x)} \right| \\ &= \ln \left| \frac{1+\sin(x)}{\cos(x)} \right| \\ &= \ln \left| \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} \right| = \ln|\sec(x) + \tan(x)|. \end{aligned}$$

**Problem 4 (Bonus).** What if we want to find  $\int \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)} dx$ ?

**Solution:** The numerator is lower degree than the denominator, so we can begin a partial fraction decomposition immediately. We set up:

$$\begin{aligned} \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+1} + \frac{Dx+E}{x^2+4x+5} \\ x^4 + 6x^3 + 4x^2 + 8x + 11 &= A(x-1)(2x+1)(x^2+4x+5) + B(2x+1)(x^2+4x+5) \\ &\quad + C(x-1)^2(x^2+4x+5) + (Dx+E)(x-1)^2(2x+1) \\ &= A(2x^4 + 7x^3 + 5x^2 - 9x - 5) + B(2x^3 + 9x^2 + 14x + 5) \\ &\quad + C(x^4 + 2x^3 - 2x^2 - 6x + 5) + D(2x^4 - 3x^3 + x) \\ &\quad + E(2x^3 - 3x^2 + 1) \\ &= (2A + C + 2D)x^4 + (7A + 2B + 2C - 3D + 2E)x^3 \\ &\quad + (5A + 9B - 2C - 3E)x^2 + (-9A + 14B - 6C + D)x \\ &\quad + (-5A + 5B + 5C + E) \end{aligned}$$

We get a system of equations

$$\begin{aligned} 2A + C + 2D &= 1 & 7A + 2B + 2C - 3D + 2E &= 6 \\ 5A + 9B - 2C - 3E &= 4 & -9A + 14B - 6C + D &= 8 \\ -5A + 5B + 5C + E &= 11. \end{aligned}$$

After solving this (admittedly nasty) collection of equations, we see that  $A = 0$ ,  $B = 1$ ,  $C = 1$ ,  $D = 0$ ,  $E = 1$ . So

$$\begin{aligned} \int \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)} dx &= \int \frac{1}{(x+1)^2} + \frac{1}{2x+1} + \frac{1}{x^2+4x+5} dx \\ &= -(x+1)^{-1} + \frac{1}{2} \ln|2x+1| + \int \frac{dx}{x^2+4x+5}. \end{aligned}$$

We complete the square, and see that  $x^2 + 4x + 5 = (x+2)^2 + 1$ , so we use  $u = x+2$  and get

$$\int \frac{dx}{x^2+4x+5} = \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

Thus

$$\int \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)} dx = -(x+1)^{-1} + \frac{1}{2} \ln|2x+1| + \arctan(x+2) + C.$$

**Problem 5.** We want to find the cross-sectional area of a two-meter-long airplane wing. We measure its width every 20 centimeters, and get: 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, 2.8. Use the trapezoidal rule and Simpson's rule to estimate the area of the wing.

**Solution:**

$$\begin{aligned}
 T_{10} &= .2 \left( \frac{5.8 + 20.3}{2} + \frac{20.3 + 26.7}{2} + \frac{26.7 + 29.0}{2} + \frac{29.0 + 27.6}{2} + \frac{27.6 + 27.3}{2} \right. \\
 &\quad \left. + \frac{27.3 + 23.8}{2} + \frac{23.8 + 20.5}{2} + \frac{20.5 + 15.1}{2} + \frac{15.1 + 8.7}{2} + \frac{8.7 + 2.8}{2} \right) \\
 &= 40.66 \\
 S_{10} &= \frac{1}{15} (5.8 + 4 \cdot 20.3 + 2 \cdot 26.7 + 4 \cdot 29.0 + 2 \cdot 27.6 + 4 \cdot 27.3 \\
 &\quad + 2 \cdot 23.8 + 4 \cdot 20.5 + 2 \cdot 15.1 + 4 \cdot 8.7 + 2.8) \\
 &= \frac{618.2}{15} = 41.2133.
 \end{aligned}$$

**Problem 6.** Consider the function  $f(x) = x^2 + 1$ .

- Use the trapezoid rule with six intervals to estimate  $\int_{-4}^2 f(x) dx$ .
- Use the midpoint rule with six intervals to estimate  $\int_{-4}^2 f(x) dx$ .
- Use Simpson's rule with six intervals to estimate  $\int_{-4}^2 f(x) dx$ .
- Which of these do you expect to be most accurate? Which do you expect to be least accurate?
- Compute  $\int_{-4}^2 f(x) dx$ . What do you find? Why?

**Solution:**

(a)

$$\begin{aligned}
 \int_{-4}^2 x^2 + 1 dx &\approx \frac{f(-4) + f(-3)}{2} + \frac{f(-3) + f(-2)}{2} + \frac{f(-2) + f(-1)}{2} \\
 &\quad + \frac{f(-1) + f(0)}{2} + \frac{f(0) + f(1)}{2} + \frac{f(1) + f(2)}{2} \\
 &= \frac{17 + 10}{2} + \frac{10 + 5}{2} + \frac{5 + 2}{2} + \frac{2 + 1}{2} + \frac{1 + 2}{2} + \frac{2 + 5}{2} \\
 &= \frac{1}{2}(27 + 15 + 7 + 3 + 3 + 7) = \frac{62}{2} = 31.
 \end{aligned}$$

Alternatively, we could write

$$\begin{aligned}
 \int_{-4}^2 x^2 + 1 dx &\approx \frac{1}{2} \left( f(-4) + 2f(-3) + 2f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2) \right) \\
 &= \frac{1}{2} (17 + 20 + 10 + 4 + 2 + 4 + 5) = \frac{1}{2} \cdot 62 = 31.
 \end{aligned}$$

(b)

$$\begin{aligned}\int_{-4}^2 x^2 + 1 \, dx &\approx f(-3.5) + f(-2.5) + f(-1.5) + f(-0.5) + f(0.5) + f(1.5) \\ &= 13.25 + 7.25 + 3.25 + 1.25 + 1.25 + 3.25 = 29.5.\end{aligned}$$

(c)

$$\begin{aligned}\int_{-4}^2 x^2 + 1 \, dx &\approx \frac{1}{3} \left( f(-4) + 4f(-3) + 2f(-2) + 4f(-1) + 2f(0) + 4f(1) + f(2) \right) \\ &= \frac{1}{3} \left( 17 + 40 + 10 + 8 + 2 + 8 + 5 \right) = 30.\end{aligned}$$

(d) We'd expect these to be roughly in increasing order of accuracy, with the trapezoid rule being the least accurate, and the midpoint rule being the most accurate.

(e)

$$\begin{aligned}\int_{-4}^2 x^2 + 1 \, dx &= \left. \frac{x^3}{3} + x \right|_{-4}^2 \\ &= \frac{8}{3} + 2 - \left( \frac{-64}{3} - 4 \right) \\ &= \frac{72}{3} + 6 = 24 + 6 = 30.\end{aligned}$$

Simpson's rule got it exactly correct! This makes sense, because the fourth derivative is zero.

**Problem 7.** Let  $g(x) = e^{-x^2}$ , and suppose we want to compute  $\int_{-1}^2 e^{-x^2} \, dx$ , and get the answer correct to two decimal places.

- We can compute that  $g''(x)$  varies between  $-2$  and  $.9$  when  $x$  is in  $[-1, 2]$ . What value should we take for  $K$ ?
- How many subintervals should we use to get the answer correct to within two decimal places using the trapezoid rule?
- How many subintervals should we use to get the answer correct to within two decimal places using the midpoint rule?
- We can compute that  $g'''(x)$  varies between  $-8$  and  $12$ . What value should we take for  $L$ ?
- How many subintervals should we use to get the answer correct to within two decimal places using Simpson's rule?

**Solution:**

- (a) Since  $g''(x)$  varies between  $-2$  and  $.9$ , the largest the absolute value can get will be  $2$ . Thus we should take  $K = 2$ .

If we wanted to compute this for ourselves, we could observe that

$$g''(x) = 2e^{-x^2}(2x^2 - 1)$$

$$g'''(x) = -4xe^{-x^2}(2x^2 - 3).$$

We want to maximize  $g''$  so we look for the zeroes of  $g'''$ , which happen at  $0$  and at  $\pm\sqrt{3/2}$ . We can ignore  $-\sqrt{3/2}$  since it's not in  $[-1, 2]$ , but we do need to check the endpoints, so we compute

$$g''(-1) \approx .74 \qquad g''(0) = -2$$

$$g''(\sqrt{3/2}) \approx .89 \qquad g''(2) \approx .26.$$

Thus the absolute minimum is  $-2$  and the absolute max is about  $.89$ .

- (b) The error in the trapezoid rule is

$$|E_T| \leq \frac{2(3)^3}{12n^2} < \frac{1}{100}$$

$$5400 < 12n^2$$

$$450 < n^2$$

$$21.2 < n$$

so we'd need at least 22 intervals.

- (c) The error in the midpoint rule is

$$|E_T| \leq \frac{2(3)^3}{24n^2} < \frac{1}{100}$$

$$5400 < 24n^2$$

$$225 < n^2$$

$$15 < n$$

so we'd need *sixteen* intervals to make sure the error was *less than*  $1/100$ , but 15 will guarantee the error is *at most*  $1/100$ .

- (d) We need to take  $L = 12$ .

(e) The error in Simpson's rule is

$$|E_S| \leq \frac{12 \cdot 3^5}{180n^4} < \frac{1}{100}$$

$$291600 < 180n^4$$

$$1620 < n^4$$

$$6.344 < n^4$$

so we'd only need five intervals.