Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 6
Due Tuesday, February 28

This week's mastery quiz has four topics. You should definitely submit S4 and S5, which are both new. If you have a 4/4 on M2—meaning you've gotten it completely right both times you've seen it—then you don't have to submit it. If you have a 2/2 on S3, you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Arc Length and Surface Area

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\displaystyle\int_{\pi/9}^{\pi/6} x\cos(3x)\,dx =$

Solution:

$$\begin{aligned}\int_{\pi/9}^{\pi/6} x\cos(3x)\,dx &= \frac{1}{3}x\sin(3x)\Big|_{\pi/9}^{\pi/6} - \int_{\pi/9}^{\pi/6}\frac{1}{3}\sin(3x)\,dx \\ &= \frac{\pi}{18}\sin(\pi/2) - \frac{\pi}{27}\sin(\pi/3) + \frac{1}{9}\cos(3x)\Big|_{\pi/9}^{\pi/6} \\ &= \frac{\pi}{18} - \frac{\pi\sqrt{3}}{54} + \frac{1}{9}\cos(\pi/2) - \frac{1}{9}\cos(\pi/3) \\ &= \frac{\pi}{18} - \frac{\pi\sqrt{3}}{54} - \frac{1}{18}.\end{aligned}$$

(b) $\displaystyle\int_{0}^{\pi/6} \sec^3(2t)\tan(2t)\,dt =$

Solution: We're going to take $u = \sec(2t)$ so that $du = 2\sec(2t)\tan(2t)\,dt$. We compute $u(0) = 1$ and $u(\pi/6) = \sec(\pi/3) = 2$. Then

$$\begin{aligned}\int_{0}^{\pi/6} \sec^3(2t)\tan(2t)\,dt &= \int_1^2 \frac{1}{2}u^2\,du \\ &= \frac{u^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.\end{aligned}$$

(c) $\displaystyle\int \frac{x^2+x+3}{x^2+2}\,dx =$

Solution: Polynomial long division gives

$$\frac{x^2+x+3}{x^2+2} = 1 + \frac{x+1}{x^2+2} = 1 + \frac{x}{x^2+2} + \frac{1}{x^2+2}$$

and therefore

$$\begin{aligned}\int \frac{x^2+x+3}{x^2+2}\,dx &= \int 1 + \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx \\ &= x + \frac{1}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan(x/\sqrt{2}) + C.\end{aligned}$$

S3: Numeric Integration

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_5^9 (x+4)^{3/2}\,dx$ to within 1/10? Use the trapezoid rule to approximate the integral with that many intervals.

(Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$f'(x)\frac{3}{2}(x+4)^{1/2} f''(x) \qquad = \frac{3}{4}(x+4)^{-1/2} = \frac{3}{4\sqrt{x+4}}$$
$$f''(5) = \frac{1}{4}$$
$$|E_M| \leq \frac{1/4 \cdot 4^3}{12 \cdot n^2} \leq \frac{1}{10}$$
$$n^2 \geq 40/3 \approx 13.3$$
$$n \geq 4$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$\int_5^9 (x+4)^{3/2}\,dx \approx \frac{\sqrt{9}^3 + \sqrt{10}^3}{2} + \frac{\sqrt{10}^3 + \sqrt{11}^3}{2} + \frac{\sqrt{11}^3 + \sqrt{12}^3}{2} + \frac{\sqrt{12}^3 + \sqrt{13}^3}{2}$$
$$\approx \frac{1}{2}9^{3/2} + 10^{3/2} + 11^{3/2} + 12^{3/2} + \frac{1}{2}13^{3/2}.$$

We can stop there, but numerically this is roughly 146.61. The true answer is approximately 146.54 so this is within the expected error bound.

(b) Suppose we have

$$g(0) = 5 \quad g(1) = 4 \quad g(2) = 7 \quad g(3) = 4 \quad g(4) = 2 \quad g(5) = 3 \quad g(6) = 5$$

Approximate $\int_0^6 g(x)\,dx$ using the Midpoint rule and using Simpson's rule.

Solution: For the midpoint rule, we have

$$M_3 = 2 \cdot g(1) + 2 \cdot g(3) + 2 \cdot g(5) = 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 3 = 22.$$

For Simpson's rule, we have

$$S_6 = \frac{1}{3}(5 + 4 \cdot 4 + 2 \cdot 7 + 4 \cdot 4 + 2 \cdot 2 + 4 \cdot 3 + 5)$$
$$= \frac{1}{3}(5 + 16 + 14 + 16 + 4 + 12 + 5) = \frac{1}{3} \cdot 72 = 24$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^\infty \frac{\ln(3x)}{x^2}\,dx =$

Solution:

$$\begin{aligned}
\int_1^\infty \frac{\ln(3x)}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln(3x)}{x^2}\,dx \\
&= \lim_{t\to\infty} \frac{-\ln(3x)}{x}\Big|_1^t - \int_1^t \frac{-1}{x^2}\,dx \\
&= \lim_{t\to\infty} \frac{-\ln(3x)}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t \\
\lim_{t\to\infty} &\frac{-\ln(3t)}{t} + \frac{\ln(3)}{1} - \frac{1}{t} + 1 \\
&= 1 + \ln(3) - \lim_{t\to\infty} \frac{\ln(t)^{\nearrow\infty}}{t_{\searrow\infty}} \\
&=^{\text{L'H}} 1 + \ln(3) - \lim_{t\to\infty} \frac{1/t^{\nearrow 0}}{1_{\searrow 1}} = 1 + \ln(3).
\end{aligned}$$

(b) Compute $\displaystyle\int_1^3 \frac{x}{x^2-1}\,dx =$

Solution:

$$\begin{aligned}
\int_1^3 \frac{x}{x^2-1}\,dx &= \lim_{t\to 1^+}\int_t^3 \frac{x}{x^2-1}\,dx \\
&= \lim_{t\to 1^+} \frac{1}{2}\ln|x^2-1|\Big|_t^3 \\
&= \lim_{t\to 1^+} \ln(8) - \ln|t^2-1| \\
&= \ln(8) - \lim_{x\to 1^+} \ln|t^2-1| = \infty.
\end{aligned}$$

So this integral doesn't converge.

S5: Arc Length and Surface Area

(a) Compute the arc length of the curve $(y-2)^3 = x^2$ between $y = 2$ and $y = 6$ for $x \geq 0$.

Solution: We have $x = (y-2)^{3/2}$, so $\frac{dx}{dy} = \frac{3}{2}(y-2)^{1/2}$ and

$$\begin{aligned} L &= \int_2^6 \sqrt{1 + \frac{9}{4}(y-2)}\, dy \\ &= \frac{8}{27}\left(1 + \frac{9}{4}(y-2)\right)^{3/2}\Bigg|_2^6 \\ &= \frac{8}{27}\left(10^{3/2} - 1\right). \end{aligned}$$

(I actually screwed up here. Technically there are two branches here, with $x = (y-2)^{3/2}$ and $x = -(y-2)^{3/2}$. So an answer of $\frac{16}{27}\left(10^{3/2} - 1\right)$ is actually more correct. But I won't deduct points from anyone who doesn't notice this.)

(b) Set up (but don't compute!) an integral for the area of a surface obtained by taking the curve $y = \ln(x^3 + 1)$ from $x = 0$ to $x = 10$ and rotating around the x axis.

Solution: We have $y' = \frac{3x^2}{x^3+1}$ so we get

$$\begin{aligned} SA &= \int_0^{10} 2\pi y \sqrt{1 + y'^2}\, dx \\ &= \int_0^{10} 2\pi \ln(x^3 + 1)\sqrt{1 + \frac{9x^4}{(x^3+1)^2}}\, dx. \end{aligned}$$