Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2023
Recitation 6

Jay Daigle

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Problem 1. We want to compute $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$.

(a) Can you compute an antiderivative? Can you evaluate it at 0 and 2?

(b) Did part (a) finish the problem? Sketch a picture of the graph. What should we be concerned about?

(c) Carefully set up a computation that will find $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$. (Hint: It should have two limit operations in it.)

(d) What did we learn from this that we didn't learn from (a)?

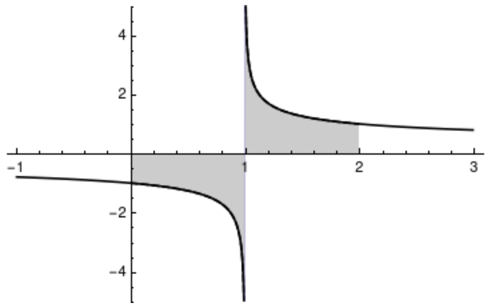
Solution:

(a) We know that

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx = \frac{3}{2}(x-1)^{2/3}$$
$$\frac{3}{2}(x-1)^{2/3}\Big|_0^2 = \frac{3}{2}(1)^{2/3} - \frac{3}{2}(-1)^{2/3} = 0.$$

(b) This is not a correct way to compute this integral, because we're assuming everything makes sense in the middle. But it does not.

This is an improper integral, because the function is discontinuous and undefined at 1.



(c) To compute, we break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_0^t - \lim_{s\to 1^+} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_s^2 \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\cdot 1\right) + \lim_{s\to 1^+} \left(\frac{3}{2}\cdot 1 - \frac{3}{2}(s-1)^{2/3}\right) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

(d) We get the "same answer" here, of 0. But this computation gives us an extra piece of information: all the area involved is finite. Saying the net area is $3/2 - 3/2$ is very different from trying to say it's $\infty - \infty$ somehow.

Problem 2. We want to figure out if $\int_0^{+\infty} e^{-x^2}\,dx$ converges—that is, if it's finite or infinite.

(a) If we can find an antiderivative, we can just compute the improper integral directly. Why doesn't that work?

(b) Since we can't integrate this directly we might want to use a comparison test. We need to find an easy-to-integrate function that's larger than e^{-x^2}. Find a function $f(x)$ that makes $f(x)e^{-x^2}$ easy to integrate.

(c) If $f(x) \geq 1$, then we can just integrate $f(x)e^{-x^2}$. Is it?

(d) This is where we can pull in a trick. Is there some a where $f(x) > 1$ when $x > a$? (You may need to adjust your $f(x)$ here, especially the sign. It's fine as long as you can still integrate it.)

(e) We know $\int_a^{+\infty} e^{-x^2}\,dx \leq \int_a^{+\infty} f(x)e^{-x^2}\,dx$. Compute the new improper integral; is it finite?

(f) Now we just have to deal with $\int_0^a e^{-x^2}\,dx$. We can't do that integral exactly, but that's fine: you should be able to tell whether it's finite or not without doing any calculations. How?

(g) Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

Solution:

(a) This is precisely the standard function we know we don't have an elementary antiderivative for. So that won't help.

(b) The obvious value for $f(x)$ is $-2x$, since that's the chain rule from e^{-x^2}. It will work out easier in the long run if we take $f(x) = 2x$, though.

(c) Clearly $-2xe^{-x^2}$ is often less than e^{-x^2}, since the first is negative and the second is positive. Even if we fix that problem, we still see that $2xe^{-x^2} < e^{-x^2}$ when $x < 1/2$.

(d) If we take $a = 1/2$, or any larger number, we fix this problem. I'm going to take $a = 1$; then we have $2xe^{-x^2} > e^{-x^2}$ for $x > 1$.

(e) In class we showed that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges. In particular we showed that

$$\begin{aligned}
\int_1^{+\infty} 2xe^{-x^2}\,dx &= \lim_{s\to+\infty} \int_1^s 2xe^{-x^2}\,dx \\
&= \lim_{s\to+\infty} \int_1^{s^2} e^{-u}\,du \\
&= \lim_{s\to+\infty} -e^{-u}\big|_1^{s^2} \\
&= \lim_{s\to+\infty} -e^{-s^2} - (-e^{-1}) = 0 + \frac{1}{e} = \frac{1}{e}.
\end{aligned}$$

So we know that

$$\int_1^{+\infty} e^{-x^2}\,dx < \int_1^{+\infty} 2xe^{-x^2}\,dx = \frac{1}{e}$$

and thus $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

(f) $\int_0^1 e^{-x^2}\,dx$ isn't an integral we can do. But it's a nice, proper integral, so nothing weird can happen. An integral of a function that's defined and continuous everywhere on the closed interval $[0, 1]$ will always *converge*. (Numerically, it works out to about 0.75.)

(g) We have concluded that

$$\begin{aligned}\int_0^{+\infty} e^{-x^2}\,dx &= \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx \\ &< \int_0^1 e^{-x^2}\,dx + \frac{1}{e} < \infty.\end{aligned}$$

In fact, this tells us that $\int_0^{+\infty} e^{-x^2}\,dx < \frac{1}{e} + .75 \approx 1.12$.

Using fancy techniques from cmoplex analysis, we can determine that $\int_0^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \approx .88$. That's way outside the scope of what we can do in this course, though.

Problem 3. Consider the graph of the hyperbola $xy = 1$ as x varies from $1/3$ to 1, and y varies from 1 to 3. We want to find the arc length of this curve.

(a) The obvious choice is to write this as a function of x. What would the function be there?

(b) Set up an integral to compute the arc length writing y as a function of x.

(c) Alternatively, we could write this as a function of y. What would that function be?

(d) Set up an integral to compute the arc length writing x as a function of y.

(e) Are these integrals the same? Can you compute them?

(f) Plug the integrals you set up into an integral calculator. Do they get the same answer?

Solution:

(a) We could view this as a function of x: $y = 1/x$.

(b) We compute that $y' = -1/x^2$, so

$$L = \int_{1/3}^1 \sqrt{1 + 1/x^4}\,dx.$$

(c) Alternatively, we could view it as a function of y: $x = 1/y$.

(d) We compute $x' = -1/y^2$, and we have

$$L = \int_1^3 \sqrt{1 + 1/y^4}\,dy.$$

(e) These integrals aren't the same. They do call for the same antiderivative, but the bounds are different.

I can't do that antiderivative. I asked Mathematica, and it tells me the integral is

$$\int \sqrt{1+1/x^4}\,dx = -\frac{\sqrt{\frac{1}{x^4}+1}x\left(x^4 - 2(-1)^{3/4}\sqrt{x^4+1}xF\left(i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle| -1\right) + 2(-1)^{3/4}\sqrt{x^4+1}xE\left(i\sinh^{-1}\left(\sqrt[4]{-1}x\right)\middle| -1\right)+1\right)}{x^4+1}$$

which doesn't help much.

(f) When I plug this into a numerical integral calculator, I get

$$\begin{aligned} L &= \int_{1/3}^{1} \sqrt{1+1/x^4}\,dx \approx 2.14662. \\ &= \int_{1}^{3} \sqrt{1+1/y^4}\,dy \approx 2.14662. \end{aligned}$$

These are the same answer, as they should be, since they're answering the same question.

Problem 4 (Gabriel's Trumpet/Infinite Paint Can)**.** Consider a trumpet-shaped container, given by taking the curve $y = 1/x$ and rotating around the x-axis, for $x \geq 1$. We're going to imagine this as a giant, oddly-shaped paint can. (See figure **??**.)

(a) First let's find the volume of this infinite object. For this we want to take cross-sections perpendicular to the x-axis. Each cross-section will be a circle. Find a formula for the radius of this circle, given the x value.

(b) Set up an integral to compute the volume. This will be an improper integral. Remember that we need to integrate an *area*, so the integrand should be the area of the circular cross-section.

(c) Compute the integral. What does that integral tell you about this object? How much paint could we pour into it?

(d) Now we want to find the surface area. We already found a radius, so we just need to set up an (improper) integral to compute the surface area.

(e) Compute that integral. What is the surface area? How much paint would we need to paint the inside of this object?

(f) Do your answers to (c) and (e) make sense together?

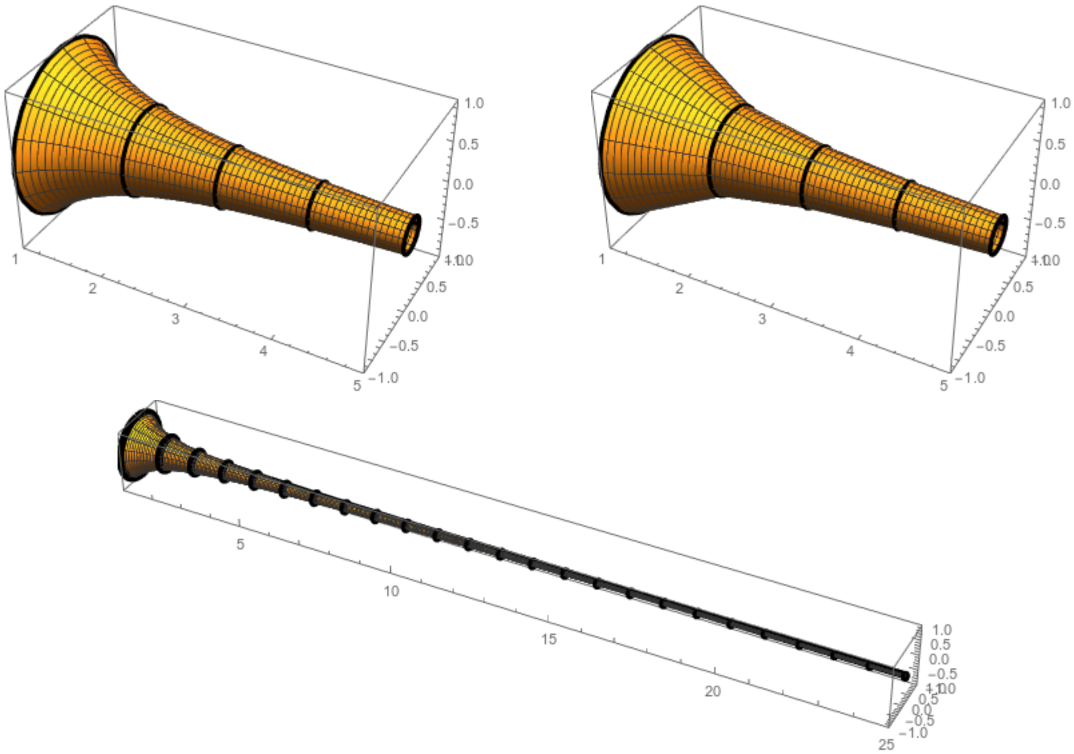


Figure 0.1: Gabriel's Trumpet

Solution:

(a) If we take cross-sections perpendicular to the x-axis, each cross section is a circle of radius $1/x$.

(b) The area of this circle will be $\frac{\pi}{x^2}$ and thus the total volume will be

$$\int_1^\infty \frac{\pi}{x^2}\,dx.$$

(c) We compute

$$\begin{aligned}\int_1^\infty \frac{\pi}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\pi}{x^2}\,dx \\ &= \lim_{t\to\infty}\frac{-\pi}{x}\Big|_1^t = \frac{-\pi}{t} - \frac{-\pi}{1} = \pi.\end{aligned}$$

Thus the volume of our paint can is π; the can can hold π gallons of paint.

(d) We can do our surface area setup. We have $f(x) = 1/x$ so that $f'(x) = -1/x^2$. Then the surface area is

$$\int_1^\infty \frac{2\pi}{x}\sqrt{1 + (-1/x^2)^2}\,dx.$$

(e) We compute

$$\begin{aligned}\int_1^\infty \frac{2\pi}{x}\sqrt{1+(-1/x^2)^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{2\pi}{x}\sqrt{1+1/x^4}\,dx \\ &\geq \lim_{t\to\infty} 2\pi \int_1^t \frac{1}{x}\,dx \\ &= 2\pi \lim_{x\to\infty} \ln|x|\Big|_1^t \\ &= \lim_{t\to\infty} 2\pi(\ln|t|-1) = \infty.\end{aligned}$$

So the surface area of the paint can is infinite!

(f) You can fill the entire can with π gallons of paint, but it would take an infinite amount of paint to cover the interior of the paint can. That doesn't really make sense, does it?

But if we actually tried to paint the trumpet, we'd have to deal with the fact that paint has *thickness* and not just surface area. Once we got far enough into the trumpet, it would be so thin that the paint wouldn't fit in at all any more.