Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 7
Due Tuesday, March 21

This week's mastery quiz has four topics. You should definitely submit S6, which is new. If you have a 4/4 on M2, then you don't have to submit it. If you have a 2/2 on S4 or S5, you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Arc Length and Surface Area
- Secondary Topic 6: Differential Equations

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) Compute $\displaystyle\int \frac{1}{(1-4x^2)^{3/2}}\,dx$.

Solution: We take $2x = \sin(\theta)$ so $2\,dx = \cos(\theta)\,d\theta$. Then we have

$$\begin{aligned}\int \frac{1}{(1-4x^2)^{3/2}}\,dx &= \int \frac{1}{(1-\sin^2(\theta))^{3/2}}\frac{1}{2}\cos(\theta)\,d\theta \\ &= \frac{1}{2}\int \frac{1}{(\cos^2(\theta))^{3/2}}\cos(\theta)\,d\theta \\ &= \frac{1}{2}\int \frac{1}{\cos^3(\theta)}\cos(\theta)\,d\theta \\ &= \frac{1}{2}\int \frac{1}{\cos^2(\theta)}\,d\theta = \frac{1}{2}\int \sec^2(\theta)\,d\theta \\ &= \frac{1}{2}\tan(\theta) + C = \frac{1}{2}\tan(\arcsin(2x)) + C.\end{aligned}$$

At this point we have a triangle with opposite side $2x$ and hypotenuse 1, so it has adjacent side $\sqrt{1-4x^2}$, which matches what we had in the integrand. Thus $\tan(\theta) = \frac{2x}{\sqrt{1-4x^2}}$ and we have

$$\int \frac{1}{(1-4x^2)^{3/2}}\,dx = \frac{x}{\sqrt{1-4x^2}} + C.$$

(b) Compute $\displaystyle\int \sin(2x)e^{5x}\,dx$.

Solution:

$$\begin{aligned}\int \sin(2x)e^{5x}\,dx &= \sin(2x)\frac{e^{5x}}{5} - \int 2\cos(2x)\frac{e^{5x}}{5}\,dx \\ &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{5}\left(\cos(2x)\frac{e^{5x}}{5} - \int -2\sin(2x)\frac{e^{5x}}{5}\,dx\right) \\ &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} + \frac{4}{25}\int \sin(2x)e^{5x}\,dx \\ \frac{29}{25}\int \sin(2x)e^{5x}\,dx &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} + C \\ \int \sin(2x)e^{5x}\,dx &= \frac{5}{29}\sin(2x)e^{5x} - \frac{2}{29}\cos(2x)e^{5x} + C.\end{aligned}$$

(c) Compute $\displaystyle\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\,dx$.

Solution: We need to do a partial fractions decomposition. We have

$$\begin{aligned}
\frac{4x^2 - x + 10}{(x-2)(x^2+4)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \\
4x^2 - x + 10 &= A(x^2+4) + (Bx+C)(x-2).2: \quad 24 &= A \cdot 8 \\
A &= 30: \quad 10 &= 4A + (C)(-2) = 12 - 2C \\
C &= 6 - 5 = 1 \\
1: \quad 13 &= 3(1+4) + (B+1)(1-2) = 15 - B - 1 \\
B &= 1.
\end{aligned}$$

Thus we compute

$$\begin{aligned}
\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\,dx &= \int \frac{3}{x-2} + \frac{x+1}{x^2+4}\,dx \\
&= \int \frac{3}{x-2} + \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx \\
&= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{4}\int \frac{1}{(x/2)^2+1}\,dx \\
&= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} =$

Solution:

$$\begin{aligned}
\int_1^2 \frac{dx}{x\ln(x)} &= \lim_{t\to 1^+} \int_t^2 \frac{dx}{x\ln(x)} \\
&= \lim_{t\to 1^+} \ln(|\ln(x)|)\Big|_t^2 \\
&= \lim_{t\to 1^+} \ln(|\ln(2)|) - \ln|\ln(t)|
\end{aligned}$$

But $\lim_{t\to 1^+}\ln(t) = 0$, so $\lim_{t\to 1^+}\ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\displaystyle\int_1^\infty \frac{2}{x^3}\,dx =$

Solution:

$$\int_1^\infty \frac{dx}{x^4} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^4}$$
$$= \lim_{t\to\infty} \frac{-1}{3x^3}\Big|_1^t$$
$$= \lim_{t\to\infty} \frac{1}{3} - \frac{1}{3t^3} = \frac{1}{3}.$$

S5: Arc Length and Surface Area

(a) Set up (but don't evaluate!) an integral that gives the arc length of the curve $\arctan(x) = y^3$ as x varies from 1 to 6.

Solution: We have $y = \sqrt[3]{\arctan(x)}$ so $y' = \frac{1}{3}(\arctan(x))^{-2/3}\frac{1}{1+x^2}$, and thus

$$L = \int_1^6 \sqrt{1 + \frac{1}{9\arctan(x)^{4/3}(1+x^2)^2}}\, dx.$$

(b) Compute (and do evaluate) the area of the surface obtained by taking the curve $y = \sqrt{15-x}$ as x goes from 3 to 5 and rotating it around the x-axis.

Solution: We have $y' = \frac{-1}{2\sqrt{15-x}}$. So we get

$$A = \int_3^5 2\pi y\sqrt{1+y'^2}\, dx$$
$$= \int_3^5 2\pi\sqrt{15-x}\sqrt{1+\frac{1}{4(15-x)}}\, dx$$
$$= 2\pi \int_3^5 \sqrt{15 - x + \frac{1}{4}}\, dx$$
$$= \pi \int_3^5 \sqrt{61-4x}\, dx$$
$$= \pi\frac{2}{3\cdot(-4)}(61-4x)^{3/2}\Big|_3^5 = \frac{-\pi}{6}\left(41^{3/2} - 49^{3/2}\right)$$
$$= \frac{\pi}{6}(343 - 41\sqrt{41}) =\approx 42.13.$$

S6: Differential Equations

(a) Find a general solution to the equation $y' = x^2 + 1 + x^2y + y$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= (x^2+1)(1+y) \\
\frac{dy}{y} &= x^2+1\,dx \\
\ln|1+y| &= {}^{x^3}\!/_3 + x + C \\
1+y &= e^{x^3/3+x+C} \\
y &= e^{x^3/3+x+C} - 1 = Ke^{x^3/3+x} - 1.
\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $-2x + 4y^3\sqrt{x^2+3}\cdot y' = 0$ if $y(1) = 1$

Solution:

$$\begin{aligned}
4y^3y'\sqrt{x^2+3} &= 2x \\
4y^3\,dy &= \frac{2x}{\sqrt{x^2+3}}\,dx \\
y^4 &= 2\sqrt{x^2+3} + C \\
y &= \sqrt[4]{2\sqrt{x^2+3}+C}.
\end{aligned}$$

Then we have

$$\begin{aligned}
1 &= \sqrt[4]{2\sqrt{4}+C} = \sqrt[4]{4+C} \\
1 &= 4+C \\
C &= -3 \\
y &= \sqrt[4]{2\sqrt{x^2+3}-3}.
\end{aligned}$$