Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 9
Due Tuesday, April 4

This week's mastery quiz has two topics. You should definitely submit M3, which are new. If you have a 2/2 on S7, you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

Analyze the convergence of the following series. (Determine if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n+1}$

Solution: We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{n+1}/5^{n+1}+1}{(-1)^n 3^n/5^n+1}\right| &= \lim_{n\to\infty}\frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)}\\ &= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1}\\ &= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.\end{aligned}$$

This limit is less than 1, so by the ratio test this converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution: This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful: $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty}\frac{n/n^2+1}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty}\frac{n}{\ln(n)} = \lim_{x\to+\infty}\frac{x^{\nearrow\infty}}{\ln(x)_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to+\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

S7: Sequences and Series

(a) Let $b_n = \frac{n!}{2^n}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$$b_1 = \frac{1}{2} \qquad b_2 = \frac{2}{4}$$
$$b_3 = \frac{6}{8} \qquad b_4 = \frac{24}{16}.$$

We see that

$$\frac{n!}{2^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{2(2)(2)\dots(2)(2)} \geq \frac{n}{2}.$$

Since $\lim_{n\to\infty} \frac{n}{2} = \infty$, we know that $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$.

(b) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

Solution: We can do a partial fractions decomposition: we have

$$\begin{aligned} 2 &= A(n+1) + B(n+3) \\ 2 &= 2B && \Rightarrow B = 1 \\ 2 &= -2A && \Rightarrow A = -1 \end{aligned}$$

so our sum is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

More rigorously, we have

$$\begin{aligned} \sum_{n=1}^{k} \frac{2}{n^2+4n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\ &\quad + \dots + \left(\frac{1}{k+1} - \frac{1}{k+3}\right) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\ \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

(c) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{2n}}$.

Solution: This is a geometric series with $a = \frac{4}{9}$ and $r = \frac{2}{9}$. Since $r < 1$ this series converges, and the sum of the series is

$$\frac{4/9}{1 - 2/9} = \frac{4/9}{7/9} = \frac{4}{7}.$$