Math 1231 Midterm Solutions

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Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a) $\lim_{x\to+\infty} \frac{\sqrt{5x^4+x+1}}{x^2+3x+4} =$

Solution:

$$\lim_{x\to+\infty} \frac{\sqrt{5x^4+x+1}}{x^2+3x+4} = \lim_{x\to+\infty} \frac{\sqrt{5+1/x^3+1/x^4}}{1+3/x+4/x^2} = \frac{\sqrt{5}}{1} = \sqrt{5}.$$

(b) $\lim_{x\to 0} \frac{\sin(3x)\tan(2x)}{5x^2} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x)\tan(2x)}{5x^2} &= \lim_{x\to 0} \frac{\frac{\sin(3x)}{3x}\cdot 3x \cdot \frac{\sin(2x)}{2x}\frac{2x}{\cos(2x)}}{5x^2} \\
&= \lim_{x\to 0} \frac{3x\cdot 2x}{5x^2\cos(2x)} \\
&= \lim_{x\to 0} \frac{6}{5\cos(2x)} = \frac{6}{5}.
\end{aligned}$$

(c) Compute $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$.

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} &= \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.
\end{aligned}$$

(d)

$$\lim_{x\to 1} \frac{x^2-4x+4}{x-1}$$

Solution:

$$\lim_{x\to 1} \frac{x^2-4x+4^{\nearrow 1}}{x-1_{\searrow 0}} = \pm\infty.$$

Problem 2 (M2)**.** Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a)

$$\frac{d}{dx}\frac{\sin^{3/7}(x^2)+x}{x^3\tan(x^3)}$$

Solution:

$$\begin{aligned}&\frac{d}{dx}\frac{\sin^{3/7}(x^2)+x}{x^3\tan(x^3)}\\ &=\frac{\left(\frac{3}{7}\sin^{-4/7}(x^2)2x+1\right)x^3\tan(x^3)-\left(3x^2\tan(x^3)+x^3 3\sec^2(x^3)3x^2\right)\left(\sin^{3/7}(x^2)+x\right)}{(x^3\tan(x^3))^2}\end{aligned}$$

(b)

$$\frac{d}{dx}\csc^3(x\sqrt{x^2+\sin(x)})=$$

Solution:

$$\begin{aligned}&\frac{d}{dx}\csc^3(x\sqrt{x^2+\sin(x)})\\ &=3\csc^2(x\sqrt{x^2+\sin(x)})\cdot\big(-\csc(x\sqrt{x^2+\sin(x)})\cot(x\sqrt{x^2+\sin(x)})\big)\\ &\quad\cdot\left(\sqrt{x^2+\sin(x)}+x\frac{1}{2}(x^2+\sin(x))^{-1/2}(2x+\cos(x))\right)\end{aligned}$$

Problem 3 (S1)**.** Suppose $f(x)=3x-5$, and we want an output of approximately 4. What input a should we aim for? Find a formula for δ in terms of ε so that if our input is $a\pm\delta$ then our output will be $4\pm\varepsilon$. Explain how you found this δ and why it should give us what we want.

Solution: We want an input of about $a=3$. We want

$$\begin{aligned}|3x-5-4|&<\varepsilon\\ |3x-9|&<\varepsilon\\ 3|x-3|&<\varepsilon\\ |x-3|&<\varepsilon/3.\end{aligned}$$

So we take $\delta=\varepsilon/3$.

Problem 4 (S2)**. Directly from the definition of derivative,** compute the derivative of $f(x)=\sqrt{x-2}$ at $a=11$.

Solution:

$$\begin{aligned}f'(11)&=\lim_{h\to 0}\frac{f(11+h)-f(11)}{h}\\ &=\lim_{h\to 0}\frac{\sqrt{9+h}-\sqrt{9}}{h}\\ &=\lim_{h\to 0}\frac{(9+h)-9}{h(\sqrt{9+h}+3)}\\ &=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{3+3}=\frac{1}{6}.\end{aligned}$$

Problem 5 (S3)**.** If $f(x)=\frac{x^2+3}{x-2}$, use a linear approximation centered at 3 to estimate $f(2.9)$.

Solution:

$$\begin{aligned}
f(3) &= \frac{12}{1} = 12 \\
f'(x) &= \frac{2x(x-2) - (x^2+3)}{(x-2)^2} \\
f'(3) &= \frac{6 \cdot 1 - 12}{1^2} = -6 \\
f(x) &\approx f(3) + f'(3)(x-3) = 12 - 6(x-3) = 30 - 6x \\
f(2.9) &\approx 12 - 6(2.9-3) = 12 - (-0.6) = 12.6.
\end{aligned}$$

Problem 6 (S4)**.** If c Calories of heat are added to a cup of water, the temperature of the cup will be $T(c) = 20 + 4c$ degrees celsius.

(a) What are the units of $T'(c)$? What does it represent physically? What would it mean if T' is b ig?

(b) Compute $T'(4)$. What is the physical interpretation of this number? What physical observation could you make to check your calculation?

Solution:

(a) The units are degrees celsius per Calorie. This represents the change in the temperature of the water when you add energy [or change the amount of energy added]. If T' is big then adding a little bit of energy will change the temperature by a lot.

(b) $T'(4) = 4$. This tells us that if the temperature is currently 20 degrees Celsius, then adding one Calorie to the cup of water will will increase the temperature by 4 degrees celsius. [Alternatively: since the derivative is constant, *every* Calorie you add will increase the temperature by 4 degrees Celsius.] We could test this by starting with a cup of water that's at 20 degrees Celsius and adding one Calorie; the temperature should rise to about 24 degrees.