Math 1231 Practice Midterm Solutions

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Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x}$$

Solution:

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} = \lim_{x \to 9} \frac{(3 - \sqrt{x})(3 + \sqrt{x})}{(9 - x)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{9 - x}{(9 - x)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{1}{3 + \sqrt{x}} = 1/6.$$

(b)

$$\lim_{x \to +\infty} \frac{3x^3 + \sqrt[3]{x}}{\sqrt{9x^6 + 2x^2 + 1} + x}$$

Solution:

$$\begin{aligned}\lim_{x \to +\infty} \frac{3x^3 + \sqrt[3]{x}}{\sqrt{9x^6 + 2x^2 + 1} + x} &= \lim_{x \to +\infty} \frac{3x^3/x^3 + \sqrt[3]{x}/x^3}{\sqrt{9x^6 + 2x^2 + 1}/(\sqrt{x^6}) + x/x^3} \\ &= \lim_{x \to +\infty} \frac{3 + x^{-8/3}}{\sqrt{9 + 2x^{-4} + x^{-6}} + x^{-2}} \\ &= \lim_{x \to +\infty} \frac{3}{\sqrt{9}} = 1.\end{aligned}$$

(c)

$$\lim_{x \to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x \to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x \to 1} \left(\frac{\sin(x-1)}{x-1} \right)^2 = \left(\lim_{x \to 1} \frac{\sin(x-1)}{x-1} \right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x \to 3} \frac{x - 5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2 (M2)**.** Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec^{3/5}\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \frac{3}{5}\sec^{-2/5}\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)\cdot \tan\left(\frac{\sqrt{x^2+1}}{x+2}\right)$$
$$\cdot \frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2)-\sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4}\cdot \frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1)-\cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 3 (S1)**.**
Suppose $f(x) = x^2 - 6x$, and we want an output of approximately -9. What input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be -9 ± 2. Justify your answer.

Solution: We want an input of about $a = 3$. Our output error will be $|x^2 - 6x + 9| = |x-3|^2$. We want this to be less than 2, so we need

$$|x-3|^2 < 2$$
$$|x-3| < \sqrt{2},$$

so we can take $\delta = \sqrt{2}$.

Problem 4 (S2)**. Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{(2+h)^2+\sqrt{2+h}-2^2-\sqrt{2}}{h} \\ &= \left(\lim_{h\to 0}\frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0}\frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\ &= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\ &= 4 + \frac{1}{2\sqrt{2}}. \end{aligned}$$

Problem 5 (S3)**.** Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$. Use it to estimate $f(1.5)$.

Solution: We calculate that $f(\pi/2) = \pi/2\sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x) + x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2) + \pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x - \pi/2) = x.$$

Thus we have

$$f(1.5) \approx 1.5.$$

(The true answer is $1.49624\ldots$.)

Problem 6 (S4)**.** Suppose that $Q(p) = 3p^2 + 10p - 100$ is the number of widgets you can buy at a price of p dollars.

(a) What are the units of $Q'(p)$? What does it represent physically? What does it mean if $Q'(p)$ is big?

(b) Calculate $Q'(10)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(a) $Q'(p)$ has units of widgets per dollar. The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). If $Q'(p)$ is big, that means that raising your price by 1 dollar gets you a lot more widgets available to buy.

(b) $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for \$10, you can get approximately seventy more widgets if you raise your price to \$11.