# Math 1231 Practice Midterm Solutions

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**Problem 1** (M3). (a) Find and classify all the critical points of $f(x) = (x-5)\sqrt[3]{x^2}$. [Note: this is quite hard but it's good practice.]

**Solution:** We compute

$$\begin{aligned} f'(x) &= \sqrt[3]{x^2} + (x-5)\frac{2}{3}x^{-1/3} = x^{2/3} + \frac{2x-10}{3\sqrt[3]{x}} \\ &= \frac{3x+2x-10}{3\sqrt[3]{x}} = 5\frac{x-2}{3\sqrt[3]{x}} \end{aligned}$$

This is equal to zero when $x = 2$ and is undefined when $x = 0$, so the two critical points are $x = 0$ and $x = 2$.

We could try to use the second derivative test here, but it won't really work. We get

$$\begin{aligned} f''(x) &= \frac{10x+10}{9x^{4/3}} \\ f''(2) &= \frac{5}{3\sqrt[3]{2}} > 0 \end{aligned}$$

so we see that $f$ has a local minimum at $x = 2$, but $f''(0)$ is undefined so it doesn't help us classify $x = 0$.

Instead we compute a chart

| | $5(x-2)$ | $\frac{1}{3\sqrt[3]{x}}$ | $f'(x)$ |
|---|---|---|---|
| $x < 0$ | $-$ | $-$ | $+$ |
| $0 < x < 2$ | $-$ | $+$ | $-$ |
| $2 < x$ | $+$ | $+$ | $+$ |

Thus we conclude that $f$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.

(b) The function $g(x) = (x^2 - 3x)\sqrt[3]{x-3}$ has absolute extrema either on $(-4, -1)$ or on $[1, 4]$. Pick one of those intervals, explain why $g$ has extrema on that interval, and find the absolute extrema.

**Solution:** We compute

$$g'(x) = (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3}.$$

This is undefined when $x = 3$, and to find the zeroes we compute

$$\begin{aligned} 0 &= (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3} \\ &= (2x-3)\sqrt[3]{x-3} + \frac{x^2-3x}{3(x-3)^{2/3}} \\ 0 &= (2x-3)\sqrt[3]{x-3}\cdot 3(x-3)^{2/3} + (x^2-3x) \\ &= (2x-3)3(x-3) + (x^2-3x) = 3(2x^2-3x-6x+9) + x^2 - 3x \\ &= 7x^2 - 30x + 27 = (7x-9)(x-3) \end{aligned}$$

which is zero when $x = 3$ or $x = 9/7$. So then we have

$$\begin{aligned} g(1) &= -2\sqrt[3]{-2} = 2\sqrt[3]{2} \\ g(9/7) &= \left(\frac{81}{49} - \frac{27}{7}\right)\sqrt[3]{-12/7} = \frac{108}{49}\sqrt[3]{12/7} \\ g(3) &= 0 \\ g(4) &= 4\cdot\sqrt[3]{1} = 4. \end{aligned}$$

All of these numbers are non-negative, so the minimum value is 0, at $x = 3$. We can see that $2\sqrt[3]{2} < 2 \cdot 2 = 4$, and maybe convince ourselves that

$$\frac{108}{49}\sqrt[3]{12/7} \approx 2 \cdot 1 < 4.$$

Thus the maximum value is 4, which occurs at 4.

**Problem 2** (S5)**.** Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

**Solution:** We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

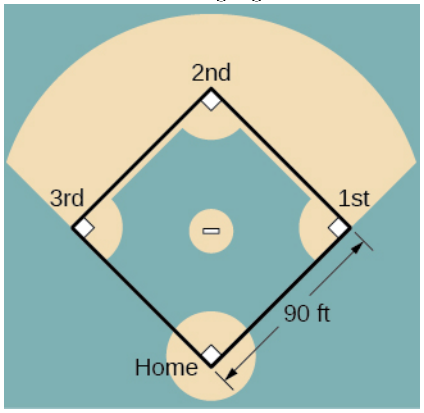
$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

**Problem 3** (S6)**.** Consider this baseball diamond, which is a square with 90ft sides. A batter hits the ball and runs from home toward first base at a speed of 22ft/s. At what rate is the distance between the runner and second base changing when the runner has run 30ft?




**Solution:** We use the Pythagorean theorem, $a^2 + b^2 = c^2$, where $a$ is the distance of the runner from first base and $b$ is the distance of second base from first base. Then $c$ is the distance between the runner and second base, which we want to know about, and we have it related to $a$ and $b$, which we do know about.

When the runner has run 30ft, then we have $a = 60$ft and $b = 90$ft is a constant. Then we have

$$c^2 = a^2 + b^2 = 60^2 + 90^2 = 3600 + 8100 = 11700$$

$$c = 10\sqrt{117} = 30\sqrt{13}.$$

Alternatively,

$$c^2 = 60^2 + 90^2 = 30^2(2^2 + 3^2)$$
$$c = 30\sqrt{13}.$$

We know that $a' = -22\text{ft/s}$, so we compute

$$\begin{aligned} 2aa' + 2bb' &= 2cc' \\ aa' + bb' &= cc' \\ 60\text{ft} \cdot (-22)\text{ft/s} + 90\text{ft} \cdot 0\text{ft/s} &= 30\sqrt{13}\text{ft} \cdot c' \\ 2 \cdot -22\sqrt{13}\text{ft/s} &= c'. \end{aligned}$$

So the distance between the runner and second base is decreasing at $\frac{44}{\sqrt{13}} \approx 12.2$ feet per second.

**Problem 4** (S7)**.** Let $f(x) = \dfrac{x^3 - 2}{x^4}$. We compute that $f'(x) = \dfrac{8 - x^3}{x^5}$ and $f''(x) = \dfrac{2x^3 - 40}{x^6}$. Sketch a graph of $f$.

Your answer should discuss the domain, asymptotes, roots, limits at infinity, critical points and values, intervals of increase and decrease, and concavity.

**Solution:** The function is defined everywhere except at 0. Near zero, we can see the top is always negative and the bottom is always positive, so $\lim_{x\to 0} f(x) = -\infty$ and we should have a downwards asymptote on either side.

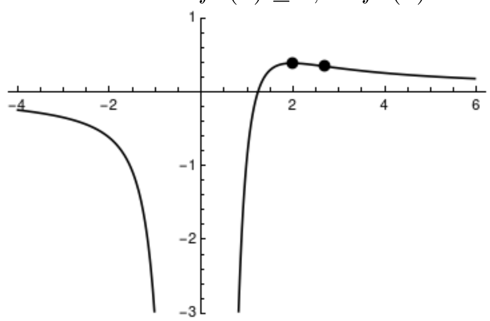
We see there is a root at $x = \sqrt[3]{2}$, and $\lim_{x\to\pm\infty} f(x) = 0$.

We see that $f'(x)$ is undefined at $x = 0$, and is zero when $x^3 = 8$ and thus when $x = 2$. So our critical points occur at 0 and 2. We calculate $f(2) = \frac{6}{16}$, and $f$ isn't defined at 0. By making a chart, we get

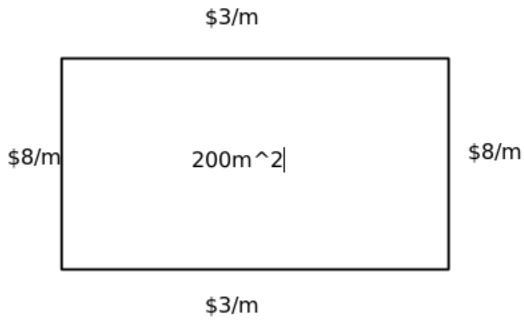
| | $8 - x^3$ | $x^5$ | $f'(x)$ |
|---|---|---|---|
| $x < 0$ | $+$ | $-$ | $-$ |
| $0 < x < 2$ | $+$ | $+$ | $+$ |
| $2 < x$ | $-$ | $+$ | $-$ |

so $f$ is decreasing for values less than zero or greater than 2, and increasing for values between 0 and 2.

The second derivative is undefined at 0, and is zero when $2x^3 - 40$ and so when $x = \sqrt[3]{20}$, so our potential points of inflection are 0, $\sqrt[3]{20}$. We compute $f(\sqrt[3]{20}) = \frac{18}{20\sqrt[3]{20}}$. We can make a chart again, but we see that the denominator of $f''(x) \geq 0$, so $f''(x) > 0$ if $x > \sqrt[3]{20}$ and $f''(x) < 0$ if $x < \sqrt[3]{20}$.




**Problem 5** (S8)**.** We want to build a rectangular fence that will enclose $200\text{m}^2$. One pair of parallel sides cost \$3/m and the other pair costs \$8/m. What dimensions minimize the cost of the fence? Justify your claim that this is a minimum.




**Solution:** We want to minimize $3w + 8\ell$ subject to the constraint $\ell w = 200$. Thus we have $w = 200/\ell$, and then our function is $C(\ell) = 600/\ell + 8\ell$. We get

$$C' = -600/\ell^2 + 8 = 0$$
$$-8\ell^2 = -600$$
$$\ell^2 = 75$$
$$\ell = 5\sqrt{3}.$$

Then we have $\ell = 5\sqrt{3}$ and $w = \frac{200}{5\sqrt{3}} = \frac{40}{3}\sqrt{3}$.

We have two options for proving this is a maximum (we only need one):

(a) Extreme Value Theorem: We can't really use the EVT here because we don't have a closed interval.

(b) First Derivative Test: For $\ell < 5\sqrt{3}$ we have $C' < 0$ so the function is decreasing, and for $\ell > 5\sqrt{3}$ we have $C' <>$ so the function is increasing. Thus we have a unique minimum at $5\sqrt{3}$.

(c) Second derivative test: $C''(\ell) = -1200/\ell^3$. Then $C''(\ell) > 0$ for $\ell ?0$, which implies there is a single relative minimum, at $\ell = 5\sqrt{3}$. This doesn't really rigorously prove that this is an absolute maximum but I'll take it.