1 Functions and Limits

1.1 Quick Review Facts

Functions

Recall that a *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input. Here we remember some facts about common functions.

Polynomials: You should remember the quadratic formula, which says that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also useful to recall that

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$
- $(a^2 + ab + b^2)(a-b) = a^3 - b^3$.

Rational functions are the ratio of two polynomials.

Trigonometric functions: In this course we will *always* use radians, because they are unitless and thus easier to track (especially when using the chain rule). Useful facts include:

- The most important trigonometric identity, and really the only one you probably need to remember, is $\cos^2(x) + \sin^2(x) = 1$.
- From this you can derive the fact that $1 + \tan^2(x) = \sec^2(x)$.
- $\sin(-x) = -\sin(x)$. We call functions like this "odd".
- $\cos(-x) = \cos(x)$. We call functions like this "even."
- $\sin(x + \pi/2) = \sin(\pi/2 - x) = \cos(x)$
- A fact that we will probably use exactly twice is the sum of angles formula for sine: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- Similarly, $\cos(x+y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

Set and interval notation

We write $\{x : \text{condition}\}$ to represent the set of all numbers x that satisfy some condition. We will sometimes write $\mathbb{R}$ to refer to all the real numbers. We will also refer to various intervals:

$(a, b) = \{x : a < x < b\}$	open interval	$[a, b] = \{x : a \leq x \leq b\}$	closed interval
$[a, b) = \{x : a \leq x < b\}$	half-open interval	$(a, b] = \{x : a < x \leq b\}$	half-open interval

1.2 Approximation

Let's start with an easy question:

Question 1.1. What is the square root of four?

Everyone can probably tell me that the answer is "two". So now let's do a harder one:

Question 1.2. What is the square root of five?

Without a calculator, you probably can't tell me the answer. But you should be able to make a pretty good guess. Five close to four; so $\sqrt{5}$ should be close to two.

We call this sort of estimate a *zeroth-order approximation*. In a zeroth-order approximation, we only get to use one piece of information: the value of our function at a specific number. Then we use that information to estimate its value at nearby numbers.

We can only do so good a job with that limited amount of information, but we can still do a surprising amount.

Example 1.3. The high temperatures in Washington DC on August 25-27 2022 were 88, 91, 88. It's not surprising that the high temperature on August 28 was 90.

Example 1.4. The high temperatures on January 1-2 were 59 and 63. Can we estimate the high on January 3?

The obvious guess is that it's roughly 60. But it turns out the actual answer is 30. Often the weather stays similar from day to day, but not always; we can't approach this question the same way we did the square root function, because some times the weather is just erratic.

This example shows that we can't always do what we did with $\sqrt{5}$. Some functions jump around too much for this sort of approximation thing to work; values of similar inputs don't have similar outputs.

And sometimes this is pretty important!

Example 1.5. Used cars that have been driven more are worth less money when sold. A 2012 study by Nicola Lacetera, Devin G. Pope, and Justin Sydnor (Heuristic Thinking and Limited Attention in the Car Market) collected data on average car price by mileage. The found (among other things) the following approximate data:

Mileage	Average Price
36000	\$11500
37000	\$11400
38000	\$11350
39000	\$11250

Based on this, what would you expect the average price of a car with 40000 miles to be?

Based just on this data it seems like you'd maybe expect a price of something like \$11200 or \$11150. But in fact the price they found was \$11000. And we see a pattern like this if we zoom out and consider all the data:

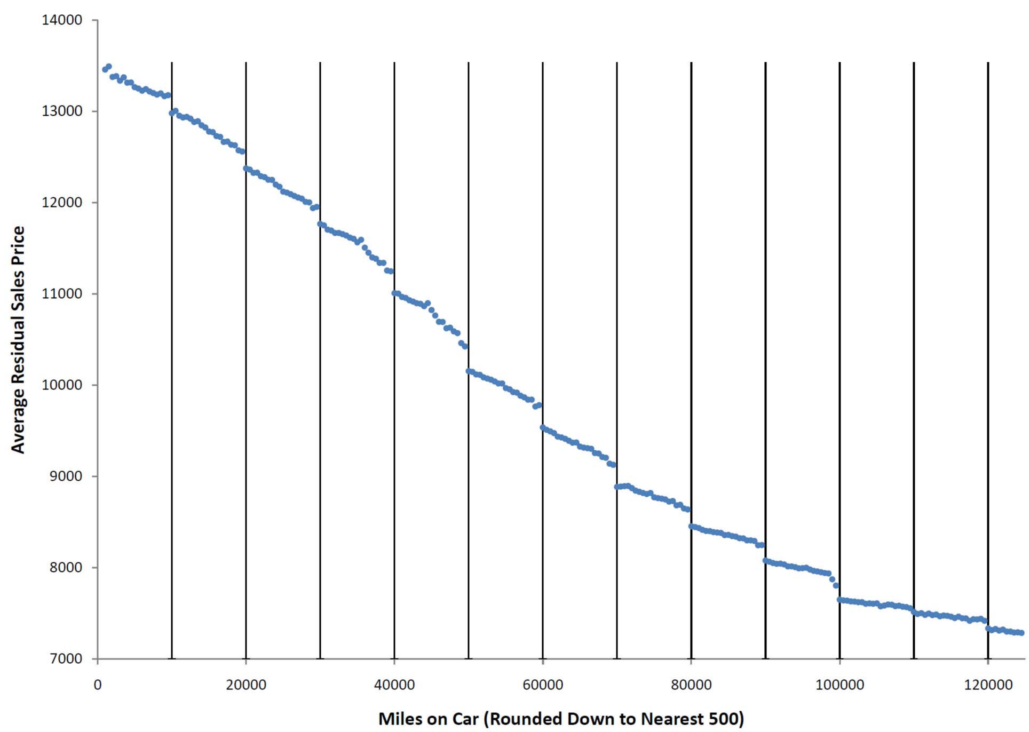


Figure 1.1: Average price of a used car at given mileages

There's a noticeable step-down every time the *first* digit of the mileage changes, because that's more noticeable to people than a change in the second or third digit. The difference between 39000 and 40000 *looks* bigger that the difference between 38000 and 3900, so there's an unexpectedly large drop in price.

On the other hand, this is all a matter of degree. The temperature dropping thirty degrees overnight is unusual, and dropping sixty is even more so; in real-world contexts we can almost always approximate things a little bit. So maybe a better question is: *how well* can we approximate a given function?

Example 1.6. Suppose I want to run a current of 5 amps, to within half of an amp, through a wire with a resistance of 2 ohms. How much voltage do I need to apply?

The formula we need is Ohm's Law, which says that $I = V/R$; in this case, we want to solve $5 = V/2$ so we get $V = 10$.

But what about the error? We want to be correct to within half an ampere, so we really want I between 4.5 and 5.5. There are a couple of ways we could write this down.

One is to say $4.5 < V/2 < 5.5$. Then we can multiply through by 2, and get $9 < V < 11$; so we need between 9 and 11 volts, or V in $(9, 11)$, or $V = 10 \pm 1$. If we want the error in our current to be less than .5, the error in our voltage has to be less than 1.

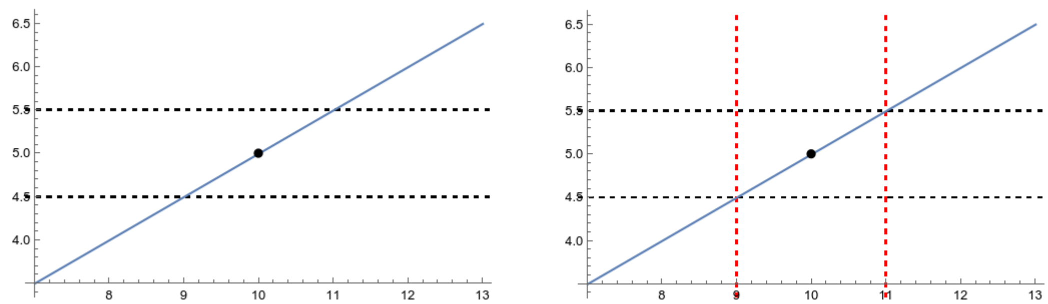


Figure 1.2: Left: We want our output to stay between the black dashed lines. Right: If our input stays between the red dashed lines, we'll hit our error threshold.

Maybe the more sophisticated way to write it is to talk about the *error*. We want the difference between our current I and our target 5 to be less than .5. We could just try writing

$$I - 5 < .5$$

but that's not quite right; it's still bad if I is too small, not just if it's too big. So instead we use the absolute value function:

$$|I - 5| < .5.$$

Since $I = V/2$, we get

$$\begin{aligned}|V/2 - 5| &< .5 \\ -.5 < V/2 - 5 &< .5 \\ 4.5 < V/2 &< 5.5 \\ 9 < V &< 11\end{aligned}$$

which gives us our original answer.

We sometimes use the Greek letter ε, which is basically the Greek lower-case "e", to represent the error in our output. So in this past example we can say that we had $\varepsilon = .5$.

What if we change the error margin? If we want $\varepsilon = .1$, we do the same setup, but instead we have

$$\begin{aligned}|V/2 - 5| &< \varepsilon = .1 \\ -.1 < V/2 - 5 &< .1 \\ 4.9 < V/2 &< 5.1 \\ 9.8 < V &< 10.2.\end{aligned}$$

So our voltage needs to be $10 \pm .2$, and the allowable error in our input voltage is .2. Just like we use the Greek letter ε for the desired error in our output, we sometimes use the Greek letter δ for the error in input. So we see here that if we want $\varepsilon = .1$ we need $\delta = .2$.

Here's a similar problem but with a more complicated function.

Example 1.7. Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

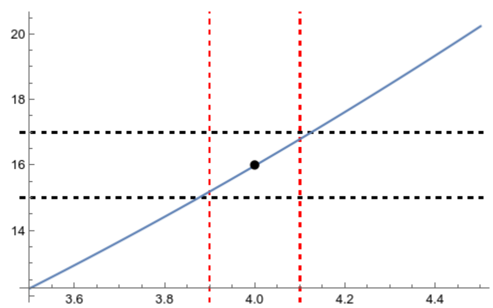
If the side length is s then the area is s^2, so we want $s^2 = 16$ or $s = 4$. (Why can't we have $s = -4$? The number -4 is also a solution to that equation, but it doesn't reflect something physically possible so we can ignore it.)

But what do we need to do to stay within our error margin $\varepsilon = 1$? Obviously, we want s to be between $\sqrt{15}$ and $\sqrt{17}$, but that's not helpful if we don't know what $\sqrt{17}$ is. Instead we're going to estimate again.

We want $|s^2 - 16| < \varepsilon = 1$, and factoring the left hand side gives $|s - 4| \cdot |s + 4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s + 4$ might be bigger than 8, but it's certainly less than 10. Then we just need to solve

$$\begin{aligned} 10|s-4| &< 1 \\ |s-4| &< .1 \\ -.1 < s-4 &< .1 \\ 3.9 < s &< 4.1. \end{aligned}$$

And indeed we can compute $3.9^2 = 15.21$ and $4.1^2 = 16.81$, both of which are within one meter of 16. We can say that we want s in the interval $(3.9, 4.1)$, or that we want s to be $4 \pm .1$. Thus we need $\delta = .1$ for our $\varepsilon = 1$.



Remark 1.8. The answer we got in example 1.7 isn't the optimal or exact bound. $4.12^2 = 16.9744 < 17$ so $\delta = .12$ still works; but this is a reasonably easy-to-compute error margin that we know will work.

In fact, the true bound doesn't have to be symmetric. If we plug in more numbers, we see that $3.875^2 = 15.0156$, which is within one square meter of 16. But $4.125^2 = 17.0156$, which gives us an error bigger than 1. We generally want to pick a δ that works in *both* directions, because we don't know whether we're going to overshoot or undershoot our targts.

Example 1.9 (recitation)**.** How does this change if we want an area of 25 square meters instead? We know we need $s \approx 5$. We want $|s^2 - 25| < 1$, which gives $|s-5| \cdot |s+5| < 1$. And here we *cannot* assume that $|s+5| < 10$, because s might be a little bigger than 10. But we can assume it's smaller than 11, so we want $s = 5 \pm 1/11$, or s in $(54/11, 56/11) \approx (4.909, 5.091)$.

And again we have $(54/11)^2 \approx 24.0992$ and $(56/11)^2 \approx 25.9174$ so $\delta = 1/11$ is in fact an acceptable amount of error in the input.

(And similarly to the last example: $4.9^2 = 24.01$ keeps us within our error margin; but $5.1^2 = 26.01$ does not. So if we want to be safe not matter which direction our error is, $\delta = .1$ is too big.)

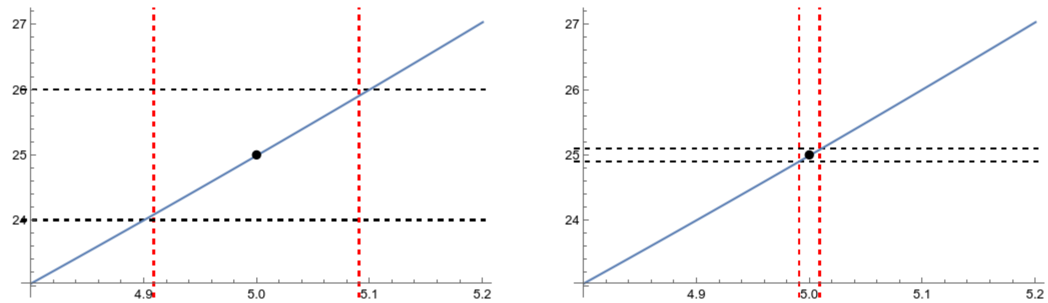


Figure 1.3: Left: $\varepsilon = 1$. Right: $\varepsilon = .1$.

Example 1.10 (recitation)**.** What if we want a smaller error, say $\varepsilon = .1$ square meters? Then we run the same calculations, and we want $|s-5| \cdot |s+5| < .1$, and thus $|s-5| < .1/11 = \frac{1}{110}$. So we want $s = 5 \pm \frac{1}{110}$, and $\delta = 1/110$.

You might be noticing at this point that if we change ε but leave everything else the same, we can generally find the new δ pretty quickly by reusing our old work.

Example 1.11 (recitation)**.** Now let's return to the square root function we started the section with. We know that $\sqrt{4} = 2$. If we want to allow an output error of $\varepsilon = 1$, how large of an input error can we tolerate? We know that $\sqrt{1} = 1$ and $\sqrt{9} = 3$, so we need to stay in the interval $(1, 9)$. Since we're aiming for 4, this means we can get away with undershooting by 3, or overshooting by 5. So we have to take $\delta = 3$.

What if we want $\varepsilon = .5$? Well, that means we want $1.5 < \sqrt{x} < 2.5$; squaring the equation gives $2.25 < x < 6.25$. Thus we can undershoot by 1.75 or overshoot by 2.25; we have to take $\delta = 1.75$.

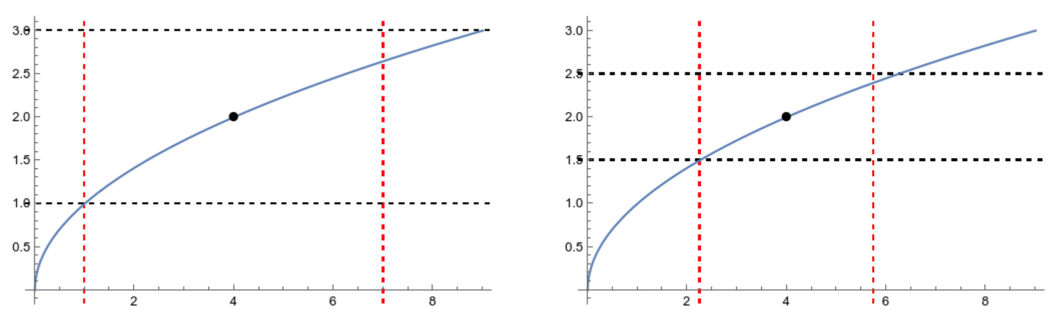


Figure 1.4: Left: $\varepsilon = 1$. Right: $\varepsilon = .5$.

Here's an example that's more complicated still—in fact, so complicated we're going to cheat:

Example 1.12 (Bonus)**.** We want to dilute an acid by a factor of ten, so that we produce a solution that's ten percent acid and 90 percent water. If we have one liter of acid, how much water do we need to add to get within one percentage point of our desired concentration?

Ideally we'd add nine liters of water, to get exactly 10% acid. But what's the error margin there, to land between 9% and 11%?

Our concentration will be $f(x) = \frac{1}{1+x}$ where x is the number of liters of water we add. We could, if we wanted, do some algebra:

$$\left|\frac{1}{1+x} - \frac{1}{10}\right| < .01$$
$$\left|\frac{10-(1+x)}{10(1+x)}\right| < .01$$
$$|9-x| < .1(1+x)$$

and since $x \approx 9$ we can assume $1+x$ is close to 10; say it's between 9 and 11. Then we need

$$|9-x| < .9$$
$$-.9 < 9-x \qquad < .9$$
$$-9.9 < -x < -8.1$$
$$8.1 < x < 9.9.$$

And we can check that if $x = 9.9$ we get a concentration of $\frac{1}{1+9.9} \approx .09174$ and if $x = 8.1$ we get a concentration of $\frac{1}{1+8.1} \approx .10989$.

But honestly, that algebra is kind of nasty. And this function isn't even that complicated, right? It gets worse. So if we really need to know the answer, we can be lazy.

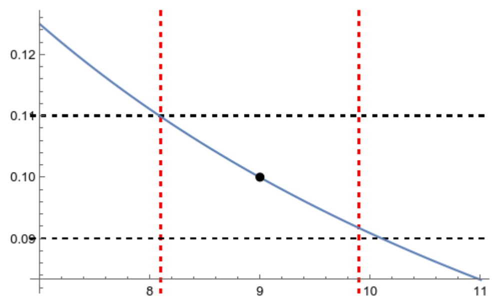
$f(9) = .1$	$f(8.9) \approx .10101$	$f(8.8) \approx .10204$
$f(8.7) \approx .10309$	$f(8.6) \approx .10417$	$f(8.5) \approx .10526$
$f(8.4) \approx .10538$	$f(8.3) \approx .10753$	$f(8.2) \approx .10870$
$f(8.1) \approx .10989$	$f(8.0) \approx .11111$	

And in the other direction

$$\begin{array}{ccc} f(9.1) \approx .09901 & f(9.2) \approx .09804 & f(9.3) \approx .09709 \\ f(9.4) \approx .09615 & f(9.5) \approx .09524 & f(9.6) \approx .09434 \\ f(9.7) \approx .09346 & f(9.8) \approx .09259 & f(9.9) \approx .09174 \\ f(10.0) \approx .09091 & f(10.1) \approx .09009 & f(10.2) \approx .08929. \end{array}$$

so we can get away with anywhere between 8.1 and 10.1 liters of water.

Then we'd have to pick $\delta = .9$, if we want to be safe in both directions.



Example 1.13. The *Heaviside function* is used to describe the behavior of a lightswitch. Before you flip the switch, no current is flowing through the circuit; when you flip the switch, current instantly jumps to 1 amp.

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

Now we're going to ask some kind-of-dumb questions.

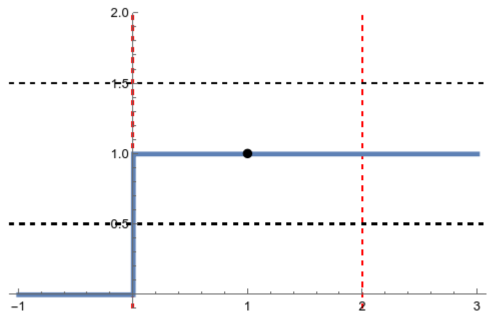
Suppose we want a current of 1 amp, plus or minus 2 amps (so $\varepsilon = 2$). What range of inputs will land us within the target range? Here we want a current between -1 amp and 3 amps, so literally any input will work.

Now let's reduce ε. What if we want a current of $1 \pm .5$? That means we want a current between .5 and 1.5. Any value $t \geq 0$ will work. But no negative value will work.

If we reduce ε still further, nothing changes. If we want a current of $1 \pm .000001$, we can still take any $t \geq 0$ and no $t < 0$.

But with just those questions, we can't really pick a δ, right? Because I didn't give you a center point. So now suppose we can take $t = 1 \pm \delta$, and we want $H(t) = 1 \pm .5$. What's the biggest δ we can possibly pick? Well, we need $t > 0$ to get $H(t) > .5$, so the biggest δ can be is 1, giving us t in $(0, 2)$.

On the other hand, what if we want $t = 0 \pm \delta$? From the definition of H, we know that $H(0) = 1$ exactly. But how far away from 0 can we get, and still land $H(t)$ in the $1 \pm .5$ target zone we're aiming for? A little thought, and maybe sketching some pictures, should convince you that we *cannot* do this. If we're aiming for $t = 0$ we cannot tolerate any error at all.



1.3 Limits

In the last subsection we talked about error margins: we have some function, and some amount of error we can accept in the output, and then we ask how much error we're allowed to make in the input. And we saw that if we picked a smaller error margin ε, we would get a smaller input tolerance δ.

But we also saw that in many of the functions we looked at, we had no problem hitting a smaller ε margin as long as we could make δ smaller at the same time.

Example 1.14. In example 1.6 we had current as a function of voltage: we got the formula $I = V/2$. And we worked out that if we want an output of $5 \pm 1/2$, we need an input of 10 ± 1.

But algebraically, we said that we want $|V/2 - 5| < \varepsilon$, and multiplying through by 2 we get $|V - 10| < 2\varepsilon$, which implies that $V = 10 \pm 2\varepsilon$ will be precise enough to serve our purposes. But we can do that whole argument without saying what ε is in the first place!

If $\varepsilon = .5$ then we get $\delta = 1$. But if $\varepsilon = .1$ we get $\delta = .2$. And if $\varepsilon = .000001$ then $\delta = .000002$. No matter *how* precise we need our output, if we make our input precise enough we can match it.

Example 1.15. In example 1.7 we were looking for the area of a square as a function of its side length. We wanted an area of 16 square meters, so we found that our error would be

$|s^2 - 16|$; after doing some tricky algebra, we saw that the condition we wanted was

$$\begin{aligned} 10|s-4| &< \varepsilon \\ |s-4| &< \varepsilon/10. \end{aligned}$$

At the time we said we wanted an error margin of 1, so we got a δ of 1/10 and wanted $s = 4 \pm 1/10$. But we can see that if we wanted to hit $\varepsilon = 1/100$ we'd just need to take $\delta = 1/1000$; we can get the area as precise as we want, as long as we can control the side length precisely.

When we have a value that we can approximate as precisely as we want, we call that a *limit*. And we can turn all the discussion we've done so far into a technical and scary-looking definition:

Definition 1.16. Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x \to a} f(x) = L,$$

if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

There's a lot of notation here, but this is just putting all the work we've been doing together. We have some target output L, and some target input a. And we have a margin of acceptable error around L, given by the number ε: we want $f(x) = L \pm \varepsilon$, or in other words we want our error $|f(x) - L| < \varepsilon$. If we can *always* hit that margin of error just by making δ small, then we say there's a limit.

Example 1.17. If $f(x) = x/2$, prove $\lim_{x \to 10} f(x) = 5$.

This is exactly what we saw in our current and voltage argument in examples 1.6 and 1.14.

Let $\varepsilon > 0$, and set $\delta = 2\varepsilon$. Then if $0 < |x - 10| < \delta$, we have

$$\begin{aligned} |f(x) - 5| = |x/2 - 5| &= \frac{1}{2}|x - 10| \\ &< \frac{1}{2}\delta = \frac{1}{2}(2\varepsilon) = \varepsilon. \end{aligned}$$

Thus $|f(x) - 5| < \varepsilon$, and so $\lim_{x \to 10} f(x) = 5$.

Example 1.18. If $f(x) = 3x$ then prove $\lim_{x \to 1} f(x) = 3$.

Let $\varepsilon > 0$ and set $\delta = \underline{\varepsilon/3}$. Then if $0 < |x - 1| < \delta$ then

$$|f(x) - 3| = |3x - 3| = 3|x - 1| < 3\delta = \varepsilon.$$

Example 1.19. If $f(x) = x^2$ then prove $\lim_{x \to 0} f(x) = 0$.

Let $\varepsilon > 0$ and set $\delta = \underline{\sqrt{\varepsilon}}$. Then if $|x - 0| < \delta$, then

$$|f(x) - 0| = |x^2| = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

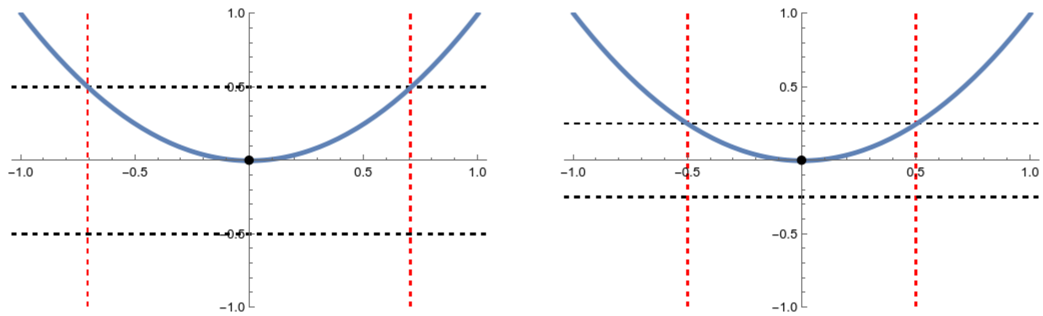


Figure 1.5: Left: $\varepsilon = 1/2$ and $\delta = 1/\sqrt{2}$. Right: $\varepsilon = 1/4$ and $\delta = 1/2$.

What does it look like when we *don't* have a limit?

Example 1.20. If $H(t)$ is the Heaviside function from example 1.13, can we compute $\lim_{t \to 0} H(t)$?

From the work we've done already, we've already seen that although $H(0) = 1$, $\lim_{t \to 0} H(t) \neq 1$. If we take $\varepsilon = .5$, we're asking all our inputs to lie between .5 and 1.5. But as long as our input is $0 \pm \delta$, then no matter how small we make δ we're still allowing negative numbers as inputs, so our outputs will include 0.

It's a little harder and more annoying to really rigorously prove that there's *no* limit, because we somehow need to talk about every possible limit at once. But just like $0 \pm \delta$ always includes negative numbers, it also always includes positive numbers, so 1 will always be an output we get. But if we ask for any number that *isn't* 1, then if ε is really small then 1 won't be between $L - \varepsilon$ and $L +$
varepsilon. We can't make epsilon as small as we like, so we don't have a limit.

However, the Heaviside function clearly behaves well if look only at one side or the other of it. And just as we could talk about continuity to one side or the other, we can talk about *one-sided limits*.

Definition 1.21. Suppose a is a real number, and f is a function which is defined for all $x < a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the left is L," and we write

$$\lim_{x \to a^-} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but less than) a.

Suppose a is a real number, and f is a function which is defined for all $x > a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the right is L," and we write

$$\lim_{x \to a^+} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but greater than) a.

Under this definition, we see that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$.

The most subtle aspect of the definition is that we don't actually care what happens when we get our input exactly right. We're only asking if we can *approximate* the desired output, not if we can get it exactly. This makes limits incredibly useful for talking about functions that are undefined at individual points.

Example 1.22. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x \to 1} f(x) = 2$.

This is harder to see at first, until we recall or notice that this function is mostly the same as $x + 1$.

Let $\varepsilon > 0$ and let $\delta = \underline{\varepsilon}$. Then if $0 < |x - 1| < \delta$, we have

$$\begin{aligned} |f(x) - 2| &= \left| \frac{x^2 - 1}{x - 1} - 2 \right| \\ &= |x + 1 - 2| && \text{since } x \neq 1 \\ &= |x - 1| < \delta = \varepsilon. \end{aligned}$$

This formal definition of limits is useful for a lot of technical work, and also for when we're trying to control the error in our output when we don't have precise control of our inputs. But it's often useful to think of it a bit more informally.

Definition 1.23 (informal)**.** Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

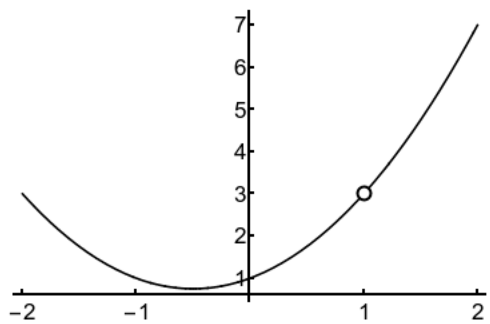
if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a, 0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Notice that we're trying to use knowing $f(x)$ to tell us what happens near a. So we specifically ignore the value of $f(a)$ even if we already know it.

Example 1.24. Let's consider the function $f(x) = \frac{x^3-1}{x-1}$. We can see the graph below. Notice that the function isn't defined at $a = 1$, so $f(1)$ is meaningless and we can't compute it.

But f is defined for all x near 1, so we can compute the limit. Looking at the graph and estimating suggests that when x gets close to 1, then $f(x)$ gets close to 3, and so we can say that $\lim_{x\to 1} f(x) = 3$.



Informally, we can estimate limits by eyeballing the graph. Formally, we can justify this limit claim by writing out a full $\varepsilon - \delta$ proof, but that's tedious and annoying. We'd like a middle route, which allows us to compute limits algebraically without having to set up a full proof; and we can do that using two core principles. The first is what I call the Almost Identical Functions property.

Lemma 1.25 (Almost Identical Functions)**.** *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.*

This tells us that two functions have the same limit at a if they have the same values near a. This makes sense, because the limit only depends on the values near a.

How does this help us? Ideally, we take a complicated function and replace it with a simpler function.

Example 1.26. Above, we looked at the function $f(x) = \frac{x^3-1}{x-1}$. You may know that we can factor the numerator; thus we in fact have $f(x) = \frac{(x-1)(x^2+x+1)}{x-1}$.

At this point you probably want to cancel the $x-1$ term on the top and the bottom. But in fact that would change the function! For $f(1)$ isn't defined. But the function $g(x) = x^2+x+1$ is perfectly well-defined at $a = 1$. Thus $f(1) \neq g(1)$, and so f and g can't be the same function.

However, they do give the same value if we plug in any number other than 1. If $y \neq 1$ then $y - 1 \neq 0$, so we have

$$f(y) = \frac{(y-1)(y^2+y+1)}{y-1} = y^2 + y + 1 = g(y).$$

Thus f and g aren't the same, but they are *almost* the same. So lemma 1.62 tells us that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$.

However, this doesn't fully answer our question. We've replaced a complicated function $f(x) = \frac{x^3-1}{x-1}$ with a simpler function $g(x) = x^2 + x + 1$, but we still haven't figured out what to do with that function. (We could still write a $\varepsilon - \delta$ proof, and we still don't want to.)

But $g(x) = x^2 + x + 1$ is a straightforward function. We can just plug numbers into it and get some sort of answer. And we would *hope* that $\lim_{x\to 1} g(x)$, which is an attempt to approximate $g(1)$, will actually give us the same answer as $g(1)$. But will it?

1.4 Continuity and Computing Limits

We began class with the observation that approximation works really well for some functions (like $\sqrt{x}$), and much less well for other functions (like the high temperature as a function of the day). We like working with the well-approximatable functions, so we give them a name: we call them *continuous*.

Informally, we say a function f is continuous at a number a if $f(x)$ is a good approximation of $f(a)$ as long as x is close to a. Formally:

Definition 1.27. We say that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

The definition of continuity says that $\lim_{x\to a} f(x) = f(a)$. This secretly actually requires three distinct things to happen:

(a) The function is defined at a; that is, a is in the domain of f.

(b) $\lim_{x\to a} f(x)$ exists.

(c) The two numbers are the same.

There are a few different ways for a function to be discontinuous at a point:

(a) A function f has a *removable discontinuity* at a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

(b) A function f has a *jump discontinuity* at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are unequal.

(c) A function f has a *infinite discontinuity* if f takes on aribtrarily large or small values near a. We'll talk about this more soon.

(d) It's also possible for the one-sided limits to not exist, but this doesn't have a special name. We'll see this with $\sin(1/x)$ when we study trigonometric functions in section 1.5. In this class, I'll just call a function like this *really bad*. But we'll mostly avoid talking about them.

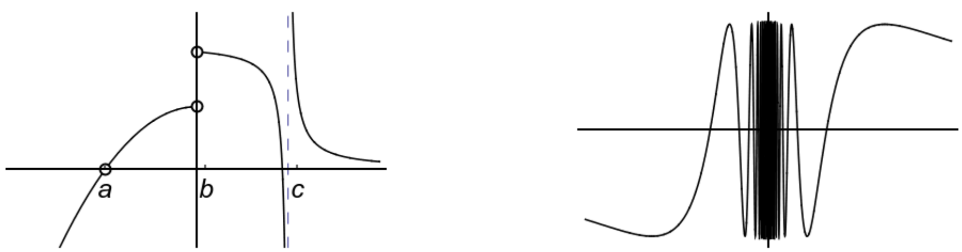


Figure 1.6: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: a "bad" discontinuity in the function $\sin(1/x)$.

Some functions get even worse than that. My two favorite discontinuous functions are:

$$T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \qquad \chi(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Example 1.28. The Heaviside function of example 1.13 is not continuous, since there's a jump at 0.

It is continuous from the right at 0, since $\lim_{x \to 0^+} H(x) = 1 = H(0)$. This function is not continuous from the left, since $\lim_{x \to 0^-} H(x) = 0 \neq H(0)$.

In fact, in some sense "most functions" aren't at all continuous. If you found away to choose $f(x)$ completely at random for each real number x, you would get a spectacularly discontinuous function. But you would never actually be able to describe it sensibly.

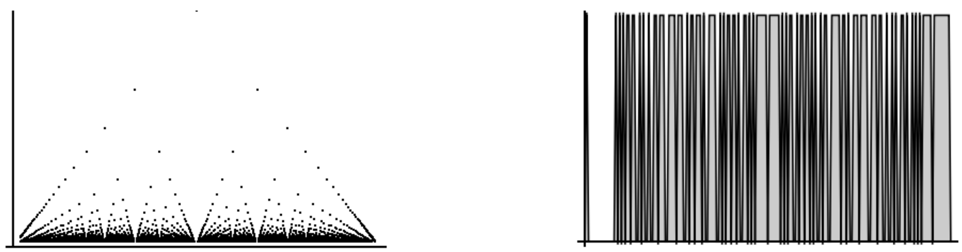


Figure 1.7: Left: $T(x)$ is really discontinuous. Right: $\chi(x)$ is really really discontinuous

In contrast, if you can draw a function reasonably, it pretty much has to be (mostly) continuous. A common informal definition is that a continuous function is one whose we can draw without lifting our pencil from the paper. Once we make this precise, this is another way to think about continuous functions. And we make it precise via the Intermediate Value Theorem

Theorem 1.29 (Intermediate Value Theorem)**.** *Suppose f is continuous (and defined!) on the closed interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$. Then there is a c in (a, b) with $f(c) = y$.*

Example 1.30. Suppose $f(x)$ is a continuous function with $f(0) = 3, f(2) = 7$. Then by the Intermediate Value Theorem there is a number c in $(0, 2)$ with $f(c) = 5$.

Example 1.31. Let $g(x) = x^3 - x + 1$. Use the Intermediate Value Theorem to show that there is a number c such that $g(c) = 4$.

To use the intermediate value theorem, we need to check that our function is continuous, and then find one input whose output is less than 4, and another whose output is greater than 4. g is a polynomial and thus continuous. Testing a few values, we see $g(0) = 1, g(1) = 1, g(2) = 7$. Since $g(1) = 1 < 4 < 7 = g(2)$, by the Intermediate Value Theorem ther is a c in $(1, 2)$ with $g(c) = 4$.

Example 1.32. Show that there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

We know that sin is a continuous function, and that $\sin(0) = 0$ and $\sin(\pi/2) = 1$. Since $0 < 1/3 < 1$, by the Intermediate Value Theorem there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

Remark 1.33. The converse of this theorem is not true. It is possible to have a function that satisfies the conclusions of the Intermediate Value Theorem, but is not continuous; these functions are called Darboux Functions.

For example, let $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then f satisfies the conclusion of the intermediate value theorem: it's continuous except at zero, so the theorem works on any interval that doesn't contain zero. Any interval containing zero contains every value in $[-1, 1]$, so if $a < 0 < b$ and y is between $f(a)$ and $f(b)$, then $-1 \leq y \leq 1$ and so there is a c in (a, b) such that $f(c) = y$. Thus f is Darboux.

Historically, the main reason we didn't take this as the definition of continuous, instead of the limit definition that we actually use, is that we didn't want to treat functions like this as "continuous".

1.4.1 Limits of Continuous Funtions

Most of the functions that we can easily describe, or draw graphs of, are continuous most of the time.

Fact 1.34. *Any reasonable function given by a reasonable single formula is continuous at any number for which it is defined.*

In particular, any function composed of algebraic operations, polynomials, exponents, and trigonometric functions is continuous at every number in its domain.

If a function is continuous at every number in its domain, we just say that it is continuous. Note, importantly, that a continuous function doesn't have to be continuous at every real number.

Example 1.35. The function

$$f(x) = \frac{x^3 - 5x + 1}{(x-1)(x-2)(x-3)}$$

is "reasonable", so it is continuous. This means that it is continuous exactly on its domain, which is $\{x : x \neq 1, 2, 3\}$.

Example 1.36. Where is $\sqrt{1 + x^3}$ continuous?

Answer: Root functions are continuous on their domains. $1 + x^3 \geq 0$ when $x \geq -1$ so the function is continuous on its domain, $[-1, +\infty)$.

Remark 1.37. Sometimes we might also talk about functions that are "continuous from the right" at a. This means that $f(a)$ is a good approximation of $f(x)$ if x is close to a and also bigger than—and thus to the right of—a.

And if we know a function is continuous, it is *very easy* to compute a limit.

Example 1.38. (a) The function $f(x) = 3x$ is continuous at 1, so $\lim_{x\to 1} f(x) = f(1) = 3$.

(b) The function $f(x) = x^2$ is continuous at 0, so $\lim_{x\to 0} f(x) = f(0) = 0$.

(c) The function $f(x) = \frac{x^2-1}{x-1}$ is definitely not continuous at 1, because it's not defined there. But we can use almost identical functions:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x + 1 = 2.$$

This might seem like the whole limit thing has no point; most functions are continuous, and if a function is continuous then we can compute limits just by plugging in values. But there is one very important type of question where limits are doing real work.

Example 1.39. What is $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$?

We use a trick called multiplication by the conjugate, which takes advantage of the fact that $(a+b)(a-b) = a^2 - b^2$. This trick is used *very often* so you should get comfortable with it.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+x}-9}{x} \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} \\
&= \lim_{x\to 0} \frac{(9+x)-3}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+3)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\lim_{x\to 0} \sqrt{9+x}+3} = \frac{1}{6}.
\end{aligned}$$

We can also use these continuity arguments to calculate one-sided limits.

Example 1.40. What is $\lim_{x\to 1^-} f(x)$ if $f(x) = \begin{cases} x^2+2 & x > 1 \\ x-3 & x < 1 \end{cases}$?

Answer: -2.

Remark 1.41. At a jump discontinuity, a function will often be continuous from one side but not the other. This is not necessarily the case, though: consider the function

$$f(x) = \begin{cases} 2 & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

Limits exist from the right and the left, but the function is not continuous from either side.

1.4.2 Function Extensions

Recall we like continuous functions because we can use their values at one point to approximate the values they should have at nearby points. And we observed that this is really unhelpful at any point where the function isn't defined. So if we have a function that's continuous everywhere it's defined, we'd like to replace it with a function that is continuous—and defined—everywhere.

Definition 1.42. We say that g is an *extension* of f if the domain of g contains the domain of f, and $g(x) = f(x)$ whenever $f(x)$ is defined.

In general, we can only extend a function to be continuous at all real numbers if the only discontinuities were removable. This is why we call discontinuities like that "removable".

Example 1.43. Let $f(x) = \frac{x^2-1}{x-1}$. Can we define a function g that agrees with f on its domain, and is continuous at all reals?

f is continuous everywhere on its domain, and is undefined at $x = 1$. We can see that $g(x) = x + 1$ will give the same value as f everywhere on f's domain, and it is continuous since it is a polynomial. Thus g is a continuous extension of f to all reals.

Alternatively, we could compute that $\lim_{x\to 1} f(x) = 2$. Then we define

$$h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1. \end{cases}$$

The function $h(x)$ is defined at all reals, and since it is continuous at 1 by our computation, it is continuous everywhere. It also must extend f since it is just defined to be f everywhere in the domain of f. So h is a continuous extension of f to all reals.

Importantly, g and h are actually the same function, since they give the same output for every input. There is at most one continuous extension of any given function; but there are multiple ways to describe that extension.

Example 1.44. The function $f(x) = 1/x$ is continuous on its domain, but we cannot extend it to a function continuous at all reals, because the limit at 0 does not exist.

Example 1.45. Let $f(x) = \frac{x^2-4x+3}{x-3}$. Can we extend f to a function continuous at all reals?

Answer: f is continuous at all reals except $x = 3$. But the function $g(x) = x - 1$ is the same everywhere except for 3, and is continuous at 3.

Example 1.46. Let

$$g(x) = \begin{cases} x^2 + 1 & x > 2 \\ 9 - 2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

Answer: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 9-2x = 5$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2+1 = 5$, so the limit at 2 exists. Thus we can extend g to

$$g_f(x) = \begin{cases} x^2+1 & x \geq 2 \\ 9-2x & x \leq 2 \end{cases}$$

which is continuous at all reals.

Example 1.47. What is $\lim_{x\to 1^-} f(x)$ if $f(x) = \begin{cases} x^2+2 & x > 1 \\ x-3 & x < 1 \end{cases}$?

Answer: -2.

1.5 Trigonometry and the Squeeze Theorem

We now want to look at limits of trigonometric functions. Fortunately, they behave *mostly* how we want them to.

Proposition 1.48. *If a is a real number, then $\lim_{x\to a}\sin(x) = \sin(a)$ and $\lim_{x\to a}\cos(x) = \cos(a)$.*

In fact, since trigonometric functions are just ways of combining sine and cosine, essentially all trigonometric functions behave this way where they are defined.

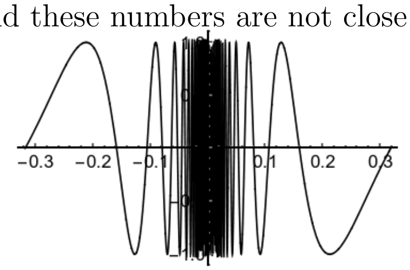
Example 1.49. $\lim_{x\to\pi}\cos(x) = -1$.

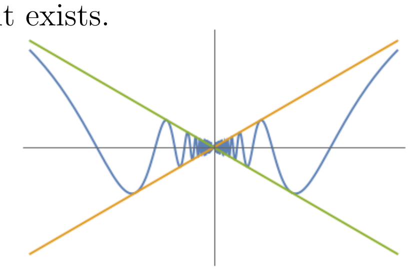
$\lim_{x\to\pi}\tan(x) = 0$.

But where the functions are not defined, sometimes very odd things can happen. We've seen a graph of $\sin(1/x)$ before, in section 1.4. We said that the function wasn't continuous at 0. In fact, no limit exists there.

Suppose a limit does exist at zero; specifically, let's suppose that $\lim_{x\to 0}\sin(1/x) = L$. Then if x is close to 0, it must be the case that $\sin(1/x)$ is close to L.

But however close we want x to be to 0, we can find a $x_1 = \frac{1}{(2n+1/2)\pi}$, and then $\sin(1/x_1) = \sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$. But we can also find an $x_2 = \frac{1}{(2n+3/2)\pi}$ so that $\sin(1/x_2) = \sin(2n\pi + 3\pi/2) = \sin(3\pi/2) = -1$. So L must be really close to 1 and really close to -1, and these numbers are not close. So no limit exists.





Left: graph of $\sin(1/x)$, Right: graph of $x\sin(1/x)$

In contrast, from the graph it appears that $\lim_{x\to 0} x\sin(1/x)$ does exist. We can't possibly prove this by replacing $x\sin(1/x)$ with an almost identical function and plugging values in: the function is gross and complicated, and any almost identical function will also be gross and complicated.

But we can easily see that $\lim_{x\to 0} x = 0$. This doesn't mean that $\lim_{x\to 0} xf(x) = 0$ for any $f(x)$; if $f(x)$ gets really big then it can "cancel out" the x term getting very small. (A good example of this is $\lim_{x\to 0} x\frac{1}{x}$, which is of course 1).

But if we can prove that the second term, which in this case is $\sin(1/x)$, does *not* get really big, then the entire limit will have to go to zero. We make this intuition precise with the following important theorem:

Theorem 1.50 (Squeeze Theorem). *If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

To use the Squeeze Theorem, we need to do two things:

(a) Find a lower bound and an upper bound for the function we're interested in; and

(b) show that their limits are equal.

We usually do this by factoring the function we care about into two pieces, where one goes to zero and the other is bounded, and thus doesn't get infinitely big.

In this case, we know that $-1 \leq \sin(a) \leq 1$ for any real number a, so in particular $-1 \leq \sin(1/x) \leq 1$. We "want" to multiply both sides of the equation by x to get $-x \leq x\sin(1/x) \leq x$, but this actually doesn't quite work! If x is negative this is in fact backwards, as we can see on the graph below:

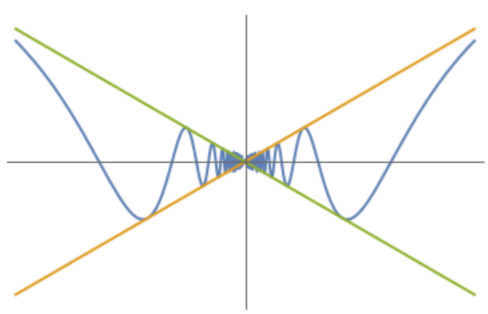
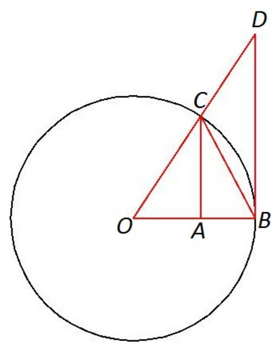


Figure 1.8: $x\sin(1/x)$ graphed with x and $-x$. Notice how they cross over in the middle.

There is one more important limit involving sin:

Proposition 1.51 (Small Angle Approximation)**.**

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



Proof. The proof of this isn't anything I'd expect you to be able to recreate, but it's interesting and fun and points to useful facts about trigonometry.

We'll assume θ is small and positive; this all still works if θ is small and negative, with different signs. We'll start by drawing the unit circle, and then including some extra lines to construct triangles.

Let θ be the measure of angle AOC in our diagram. Observe that $\sin(\theta)$ is precisely the length of the line segment AC by definition, and the line segment OC has length one because this is a unit circle. So triangle BOC has area $\frac{\sin(\theta)}{2}$.

Now we want to find the area of the wedge BOC. This will be larger than the triangle, since it includes the whole triangle plus an extra sliver. We know that the area of the entire circle is π. But the wedge is $\frac{\theta}{2\pi}$ of the circle, since the circle measures 2π radians; so the area of the wedge is $\pi \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$.

Since the triangle is contained in the wedge, we have $\frac{\sin(\theta)}{2} \leq \frac{\theta}{2}$ and thus $\frac{\sin(\theta)}{\theta} \leq 1$. That gives us one inequality involving $\frac{\sin(\theta)}{\theta}$, and now we need another.

We want to find the area of the large triangle BOD. But we know that $\tan(\theta)$ is the ratio of the opposite side of this triangle to the adjacent, so $\tan(\theta) = \frac{\mathrm{BD}}{\mathrm{OB}}$. Since OB has length 1, the length of DB is $\tan(\theta)$, and the area of this triangle is $\frac{\tan(\theta)}{2}$.

Since the wedge BOC is contained in this triangle BOD, we know that it has a smaller area, so $\frac{\theta}{2} \leq \frac{\tan(\theta)}{2}$. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, after rearranging we see that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta}$.

Thus $\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$. But $\lim_{\theta \to 0} \cos \theta = 1$, and $\lim_{\theta \to 0} 1 = 1$, so by the squeeze theorem we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

□

Remark 1.52. This means that the function

$$f(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is a continuous extension of $\sin(x)/x$ to all reals.

It also tells us that when x is a small number, $\sin(x) \approx x$. This is a fact we'll return to a couple times throughout the course.

The small angle approximation is useful on its own, but we can also use it as a new limiting rule, which allows us to compute limits of functions that mix algebra and trigonometry.

Example 1.53. Suppose we want to compute $\lim_{x\to 0} \frac{\sin(2x)}{2x}$. If we take $\theta = 2x$, then this is $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

Example 1.54. What is $\lim_{x\to 3} \frac{\sin(x-3)}{x-3}$?

This is a small angle approximation again, since we can take $\theta = x - 3$, which is approaching zero. Thus the limit is 1.

In general, we can use the small angle approximation plus a bit of algebra to work out all sorts of these computations.

Example 1.55. What is $\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x}$?

We can write

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x} &= \lim_{x\to 0} \frac{\left(\sin(4x)/4x \cdot 4x\right)\left(\sin(6x)/6x \cdot 6x\right)}{\left(\sin(2x)/2x \cdot 2x\right)\cdot x} \\ &= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 6x}{6x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{24x^2}{2x^2} \\ &= 1 \cdot 1 \cdot 1 \cdot 12 = 12. \end{aligned}$$

Here we are simply pairing off the $\sin(\theta)$'s with θs, for $\theta = 4x, 6x, 2x$, and then collecting the remainder into the last term.

Example 1.56 (Bonus)**.** What is $\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)}$?

When we see a tangent in a problem, it is often helpful to rewrite it in terms of sin and cos. We can then collect terms:

$$\begin{aligned} \lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)} &= \lim_{x\to 0} \frac{x\sin(2x)}{\sin(3x)/\cos(3x)} \\ &= \lim_{x\to 0} \frac{3x}{\sin(3x)} \cdot \frac{\sin(2x)\cos(3x)}{3} = 1 \cdot \frac{0}{3} = 0. \end{aligned}$$

Example 1.57 (Recitation)**.** What is $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$?

We have a sin(0) on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} = \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

Example 1.58. What is $\lim_{x\to 0} \frac{1-\cos x}{x}$?

We can see that the limits of the top and the bottom are both 0, so this is an indeterminate form. We can't use the small angle approximation directly because there is no sin here at all. But we can fix that by multiplying by the conjugate.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$
$$= \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0.$$

1.6 Infinite Limits

A few times in the past couple sections we've talked about vertical asymptotes, or functions going to infinity. In this section we want to look at exactly what that means. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 1.59. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

1.6.1 Limits To Infinity

Definition 1.60. We write

$$\lim_{x\to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x\to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x\to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Remark 1.61. Important note: If the limit of a function is infinity, the limit *does not exist.* This is utterly terrible English but I didn't make it up so I can't fix it. All the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Lemma 1.62. *Let $f(x), g(x)$ be defined near a, such that $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \pm\infty.$$

Further, assuming $c > 0$ then the limit is $+\infty$ if and only if $g(x) \geq 0$ near a, and the limit is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$ then the opposite is true.

Remark 1.63. If the limit of the numerator is zero, then this lemma is *not useful.* That is one of the "indeterminate forms" which requires more analysis before we can compute the limit completely.

Example 1.64. What is $\lim_{x\to 3} \frac{-1}{\sqrt{x-3}}$? We see the top goes to 1 and the bottom goes to 0, so the limit is $\pm\infty$. Since the denominator is always positive and the numerator is negative, the limit is $-\infty$.

We have to be careful while working these problems: the limit laws that work for finite limits don't always work here, since the limit laws assume that the limits exist, and these do not. In particular, adding and subtracting infinity *does not work.* In recitation we'll experiment a bit with the ways that normal limit laws fail. Instead, we need to arrange the function into a form where we can use lemma 1.62.

It's helpful organize our thinking about these situations in terms of the "indeterminate forms", which are: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty\pm\infty, 1^\infty, \infty^0$. Notice that none of these are actual numbers, and they can never be the correct answer to pretty much any question.

More importantly, indeterminate forms don't even tell us what the answer should be; if plugging in gives you one of those forms, the true limit could potentially be pretty much anything. We have to do more work to get our functional expression into a determinate form. As a general rule, we use algebraic manipulations to get a form of $\frac{0}{0}$, then factor out and cancel $(x-a)$ until either the numerator or the denominator is no longer 0.

In contrast, neither $\frac{0}{1}$ nor $\frac{1}{0}$ is an indeterminate form. $\frac{0}{1}$ is just a number, equal to 0. $\frac{1}{0}$ is not a number and is never the correct answer to a question; if you ever tell me anything is equal to $\frac{1}{0}$, in basically any context, you're not getting full credit. But it's also not indeterminate. The whole point of 1.62 is that this must be $\pm\infty$.

Similarly, $\frac{0}{\infty}$ and $\frac{\infty}{0}$ are also not numbers but not indeterminate. The first suggests the limit is 0; the second suggests the limit is $\pm\infty$.

Remark 1.65. The form $\infty \cdot \infty$ mostly works fine, and gives you another ∞ whose sign depends on the signs of the ∞s you're multiplying. But again, $\infty \cdot \infty$ is never the actual answer to any actual question.

Example 1.66. What is $\lim_{x\to 3^+} \frac{1}{(x-3)^3}$? This is *not* an indeterminate form. the limit of the top is 1, and the limit of the bottom is 0, so the limit is $\pm\infty$. But when $x > 3$ the denominator is ≥ 0, so the limit is in fact $+\infty$. Conversely $\lim_{x\to 3^-} \frac{1}{(x-3)^3} = -\infty$ since when $x < 3$ we have $(x-3)^3 \leq 0$.

$\lim_{x\to -1^+} \frac{1}{(x+1)^4} = +\infty$. And $\lim_{x\to -1^-} \frac{1}{(x+1)^4} = +\infty$. Thus $\lim_{x\to -1} \frac{1}{(x+1)^4} = +\infty$.

Example 1.67. What is $\lim_{x\to -2} \frac{1}{x+2} + \frac{2}{x(x+2)}$? This looks like $\infty + \infty$ so it is an indeterminate form, and we have to be careful. We have

$$\begin{aligned}\lim_{x\to -2} \frac{1}{x+2} + \frac{2}{x(x+2)} &= \lim_{x\to -2} \frac{x}{x(x+2)} + \frac{2}{x(x+2)} \\ &= \lim_{x\to -2} \frac{x+2}{x(x+2)} = \lim_{x\to -2} \frac{1}{x} = \frac{-1}{2}.\end{aligned}$$

1.6.2 Limits at infinity

A related concept is the idea of limits "at" infinity, which answers the question "what happens to $f(x)$ when x gets very big?"

Example 1.68.

In principle, we want to do the same thing we did for finite limits, where we find an almost identical function that's continuous, and then plug in a value of x. But we can't actually "plug in" infinity, because it's not a number, so instead we use the following rule:

Fact 1.69. $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.

This combined with tools we already have is enough to do pretty much any calculation we might want

Example 1.70. If we want to calculuate $\lim_{x\to+\infty} \frac{1}{\sqrt{x}}$, we see that

$$\lim_{x\to+\infty} \frac{1}{\sqrt{x}} = \sqrt{\lim_{x\to+\infty} \frac{1}{x}} = \sqrt{0} = 0.$$

Example 1.71. What is $\lim_{x\to+\infty} \frac{x}{x^2+1}$?

This problem illustrates the primary technique we'll use to solve infinite limits problems. It's difficult to deal with problems that have variables in the numerator and denominator, so we want to get rid of at least one. Thus we will divide out by xs on the top and the bottom until one has none left:

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x/x}{x^2/x + 1/x} = \lim_{x\to+\infty} \frac{1}{x+\frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Example 1.72 (recitation/bonus)**.** Some more examples of this technique:

$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$
$$\lim_{x\to-\infty} \frac{x}{3x+1} = \lim_{x\to-\infty} \frac{1}{3+\frac{1}{x}} = \frac{1}{3}.$$

Example 1.73. What is $\lim_{x\to+\infty} \dfrac{x^{3/2}}{\sqrt{9x^3+1}}$?

This one is a bit tricky. We want to get the xs out of the numerator, so we divide the top and bottom by $x^{3/2}$.

$$\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9x^3+1}/x^{3/2}}.$$

Then we can observe that $x^{3/2} = \sqrt{x^3}$, and so we have

$$\lim_{x\to+\infty} \frac{1}{\sqrt{9x^3+1}/\sqrt{x^3}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9+1/x^3}} = \frac{1}{\sqrt{9+0}} = \frac{1}{3}.$$

Example 1.74. Sometimes it's a bit harder to see how this works. For instance, what is $\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}$? It's not obvious, but since we can say that $x = \sqrt{x^2}$ we can use the same technique:

$$\begin{aligned}
\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to+\infty} \frac{x/x}{\sqrt{x^2+1}/x} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{x^2/x^2+1/x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.
\end{aligned}$$

But something subtle happened there that isn't obvious. It becomes important in problems that tweak things slightly.

Example 1.75. What is $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$?

We can do the same thing, but we have to be *very careful*. We know that $\sqrt{x^2}$ is always positive, so if $x < 0$ then $\sqrt{x^2} \neq x$! Instead, $x = -\sqrt{x^2}$. Thus we have

$$\begin{aligned}\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/x} \\ &= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/(-\sqrt{x^2})} \\ &= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1.\end{aligned}$$

When we encounter new functions, one of the ways we will often want to characterize them is by computing their limits at $\pm\infty$. Sometimes these limits do not exist.

Example 1.76. $\lim_{x\to+\infty} \sin(x)$ does not exist, since the function oscillates rather than settling down to one limit value.

$\lim_{x\to+\infty} x\sin(x)$ also does not exist; this function oscillates more and more wildly as x increases.

But $\lim_{x\to+\infty} \frac{1}{x}\sin(x)$ does in fact exist. We can prove this with the squeeze theorem: we can see that $\frac{-1}{x} \leq \frac{1}{x}\sin(x) \leq \frac{1}{x}$, and we know that $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$. So by the Squeeze Theorem, $\lim_{x\to+\infty} \frac{1}{x}\sin(x) = 0$.

Another technique that will also often appear in these limits is combining a sum or difference into one fraction. If we have a sum of two terms that both have infinite limits, we need to combine or factor them into one term to see what is happening.

Example 1.77. What is $\lim_{x\to-\infty} x - x^3$?

Each term goes to $-\infty$, so this is a difference of infinities and thus indeterminate. But we can factor: $\lim_{x\to-\infty} x(1-x^2)$. The first term goes to $-\infty$ and the second term also goes to $-\infty$, so we expect that their product will go to $+\infty$. Thus $\lim_{x\to-\infty} x - x^3 = +\infty$.

To be precise, I should compute:

$$\lim_{x\to-\infty} x - x^3 = \lim_{x\to-\infty} \frac{x-x^3}{1} = \lim_{x\to-\infty} \frac{1/x^2 - 1}{1/x^3}.$$

We see the limit of the top is -1 and the limit of the bottom is 0, so the limit of the whole is $\pm\infty$. In fact the bottom will always be negative (since $x \to -\infty$), and thus the limit is $+\infty$.

Example 1.78. What is $\lim_{x\to+\infty} \sqrt{x^2+1} - x$?

We might want to try to use limit laws here, but we would get $+\infty - +\infty$ which is not defined (and is one of the classic indeterminate forms). Instead we need to combine our expressions into one big fraction.

$$\begin{aligned}\lim_{x \to +\infty} \sqrt{x^2+1} - x &= \lim_{x \to +\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\ &= \lim_{x \to +\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1}+x} = \lim_{x \to +\infty} \frac{1}{\sqrt{x^2+1}+x} \\ &= \lim_{x \to +\infty} \frac{1/x}{\sqrt{1+1/x^2}+1} = 0.\end{aligned}$$

This tells us that as x increases, x and $\sqrt{x^2+1}$ get as close together as we wish.

You may have noticed the appearance of our old friend, multiplication by the conjugate. We will often use that technique in this sort of problem.