Math 1231: Single-Variable Calculus 1
George Washington University Fall 2024
Recitation 1

Jay Daigle

September 3, 2024

In class we talked about estimating functions and controlling their error margins. We had a target output, and an allowed error margin ε. Then we wanted to find the error δ we could permit in the input to keep our output from getting too far off.

Problem 1 (Warmup)**.** A cylindrical water tank has a base with an area of six square inches. We want to fill it with water ten inches deep, to the nearest inch.

(a) What is the exact volume of water would we ideally want to pour in?

(b) What ε do we want for our error margin?

(c) What possible volumes of water will allow us to stay within our error margin?

(d) What δ does that give us?

Solution:

(a) 60 cubic inches.

(b) I expect some people to think $\varepsilon = 1$ but actually $\varepsilon = .5$ here—because of the way rounding works, you want the depth to be between 9.5 and 10.5.

(c) $V = 6A$, so our volume should be between 57 and 63 cubic inches.

(d) $\delta = 3$.

Problem 2. In class we talked about the square root function. We know that $\sqrt{4} = 2$. Suppose we take $\varepsilon = 1$ so we want our output to be 2 ± 1.

(a) What is the largest input that keeps the output within ε of 2?

(b) What is the smallest input that keeps the output within ε of 2?

(c) What does that make δ?

Now instead let's take $\epsilon = .5$.

(d) What is the largest input that keeps the output within ε of 2?

(e) What is the smallest input that keeps the output within ε of 2?

(f) What does that make δ?

Solution:

(a) 9

(b) 1

(c) This is actually kind of a trick question. You *could* say that the inputs need to be 5 ± 4 so $\delta = 4$, but that's not how we want to think about it in this course. (Anyone who came up with that answer isn't really wrong! We didn't talk about this clearly in class. They're just answering a slightly different question.)

Since the input that gives us the *exact* answer we want is 4, we're looking for $4 \pm \delta$. And then we need to take $\delta = 3$; we could overshoot by 5, but we can only undershoot by 3 and still hit the target.

(You could imagine, in a real-world process, choosing to aim for 5. You'd accept being too high on average in exchange for it being easier to stay within the error margin overall. But that's not how we want to set this up.)

(d) 6.25

(e) 2.25

(f) We're looking at 4 ± 1.75 so $\delta = 1.75$. Again, you "could" take 4.25 ± 2, but that's answering a slightly different question.

In class, we looked at the following question: Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

Clearly, our sides need to be between $\sqrt{15}$ and $\sqrt{17}$ but that doesn't tell us anything useful. So instead we made the following argument: We can use an absolute value to describe the way we think about errors. in particular, what we want here is

$$|s^2 - 16| < \varepsilon = 1, \tag{1}$$

and factoring the left hand side gives $|s-4| \cdot |s+4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s + 4$ might be bigger than 8, but it's certainly less than 9, or 10. Then we just need to solve

$$|s-4| \cdot |s+4| < 10|s-4| < 1 \tag{2}$$

$$|s-4| < .1 \tag{3}$$

$$-.1 < s - 4 < .1 \tag{4}$$

$$3.9 < s - 4 < 4.1. \tag{5}$$

Thus $\delta = .1$ and s should be $4 \pm .1$.

This is a tricky argument! But I want you to try to think through it now.

Problem 3. Let's suppose instead we want to make a square platform with area 25 square meters, plus or minus 1.

(a) Write down the analogue of inequality (1) for this new problem. Can you explain in words what this inequality says about your error?

(b) We can factor the left-hand side of this inequality into two factors. If our input is close to 5, one of these terms will be small, and the other will be large. Which one will be large, and about how large will that be?

(c) This should let you write down an inequality like the one in (2). What is it?

(d) Figure out δ such that $s = 5 \pm \delta$ will keep us in our error bounds.

(e) Check your answer: square $5 + \delta$ and $5 - \delta$ and see whether the answers fall within your error margin.

(f) Could you use a larger δ than the one you found in part (4)?

Solution:

(a) $|s^2 - 25| < 1$. This says that the error between our output s^2 and our target 25 is less than one.

(b) We get $|s - 5| \cdot |s + 5| < 1$. The $|s - 5|$ term should be small since we want s close to 5; the $|s + 5|$ term will be large, and it should be approximately 10 since $s \approx 5$.

(c) $s + 5 \approx 10$ so we can say $s + 5 < 11$. So we get $|s - 5| \cdot |s + 5| < 11|s - 5| < 1$.

(d) Then we need $|s - 5| < 1/11$ which gives us $\delta = 1/11$.

(e) $(54/11)^2 \approx 24.0992$ and $(56/11)^2 \approx 25.9174$ so $\delta = 1/11$ is in fact an acceptable amount of error in the input.

(f) We see that $4.9^2 = 24.01$ keeps us within our error margin; but $5.1^2 = 26.01$ does not. So $\delta = 1/10$ is too big. However, we could take something like $\delta = .095$, which is bigger than $1/11 \approx .091$. Then $4.905^2 = 24.059$ and $5.095^2 = 25.959$ both stay within our error margin.

The largest *possible* δ that works is $\sqrt{26} - 5 \approx .099$. But it's hard to figure that out without already knowing the value of $\sqrt{26}$.

Problem 4. If time permits: redo 3 with $\varepsilon = .1$. You'll notice that you can do this pretty quickly, since you already did the hard part. If we change ε again, it should be easy to find a new δ.

Problem 5. Let $f(x) = 5x + 2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \to 2} f(x)$.

(a) If x is about 2, what should $f(x)$ be?

(b) Write down expressions using absolute value for the input and output errors.

(c) If we want $\varepsilon = 1$, what does δ need to be?

(d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

(e) Try to write a full proof.

Solution:

(a) $f(x) \approx 12$.

(b) Output error is $|f(x) - 12|$ or $|5x + 2 - 12|$, which we can simplify to $|5x - 10|$. Input error is $|x - 2|$.

(c) We want $|5x - 10| < 1$, and dividing by 5 gives $|x - 2| < 1/5$. So we'd need $\delta = 1/5$.

(d) We want $|5x - 10| < \varepsilon$, and dividing by 5 gives $|x - 2| < \varepsilon/5$. So we'd need $\delta = \varepsilon/5$.

(e) Let $\varepsilon > 0$ and set $\delta = \varepsilon/5$. Then if $0 < |x - 2| < \delta = \varepsilon/5$ we compute that

$$|f(x) - 12| = |5x - 10| = 5|x - 2| < 5 \cdot \varepsilon/5 = \varepsilon.$$

Problem 6. Let $g(x) = x^2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x\to 0} g(x)$.

(a) If x is about 0, what should $g(x)$ be?

(b) Write down expressions using absolute value for the input and output errors.

(c) If we want $\varepsilon = 1$, what does δ need to be? What about $\varepsilon = 1/4$?

(d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

(e) Try to write a full proof.

Solution:

(a) $g(x) \approx 0$.

(b) Output error is $|g(x) - 0|$ or $|x^2 - 0|$, which we can simplify to x^2. Input error is $|x - 0|$, which we can simplify to $|x|$.

(c) We want $x^2 < 1$, and taking square roots gives $|x| < 1$, so we need $\delta = 1$.

If we want $x^2 < 1/4$ then taking square roots gives $|x| < 1/2$, so we need $\delta = 1/2$.

Note that in both cases the absolute value matters; the square root of x^2 is always positive, and thus equals $|x|$.

(d) We want $x^2 < \varepsilon$, and taking square roots gives $|x| < \sqrt{\varepsilon}$. So we take $\delta = \sqrt{\varepsilon}$.

(e) Let $\varepsilon > 0$ and set $\delta = \sqrt{\varepsilon}$. Then if $0 < |x| < \delta = \sqrt{\varepsilon}$ we compute that

$$|g(x) - 0| = x^2 = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$