Math 1231 Fall 2024
Single-Variable Calculus I Section 12
Mastery Quiz 10
Due Wednesday, November 6

This week's mastery quiz has three topics. They are the same topics as last week. If you have a 4/4 in M3, or a 2/2 in S7 or S8, you don't need to submit them.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Secondary Topic 7: Curve Sketching
- Secondary Topic 8: Physical Optimization

Name:

Recitation Section:

Major Topic 3: Optimization

(a) The function $f(x) = \dfrac{x}{x^2+1}$ has absolute extrema either on the interval $[0, 3]$ or on the interval $(2, 4)$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on a closed interval, so it must have an absolute max and min by the EVT.

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

The denominator is never zero; the numerator is zero for $x = \pm 1$ so the critical points are ± 1. But -1 is irrelevant. We compute

$$f(0) = 0$$
$$f(1) = \frac{1}{2}$$
$$f(3) = \frac{3}{10}.$$

Thus the absolute maximum is $1/2$, at 1; and the absolute minimum is 0, at 0.

(b) Classify the critical points and relative extrema of $h(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$.

Solution: We have

$$h'(x) = \cos(x) - \sin(x)$$

so $h'(x)$ is defined everywhere, and has critical points where $\cos(x) = \sin(x)$. This happens when $x = \pi/4, 5\pi/4, 9\pi/4, \cdots = \pi/4 + n\pi$. We only need to care about $\pi/4$ and $5\pi/4$.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$h''(x) = -\sin(x) - \cos(x)$$
$$h''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$$
$$h''(5\pi/4) = \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} > 0.$$

Thus h has a local maximum at $\pi/4$ and has a local minimum at $5\pi/4$.

For the first derivative test we make a chart:

	$h'(x)$
$0 < x < \pi/4$	$+$
$\pi/4 < x < 5\pi/4$	$-$
$5\pi/4 < x < 2\pi$	$+$

so h has a relative maximum at $\pi/4$ and a relative minimum at $5\pi/4$.

Secondary Topic 7: Curve Sketching

Sketch the graph of $g(x) = 3x^4 - 4x^3 - 36x^2 + 64 = (x+2)^2(3x-4)(x-4)$. We have $g'(x) = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$ and $g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$.

You should discuss the domain, limits, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution:

(i) The domain is all reals.

(ii) The roots are at $x = -2, 4/3, 4$.

(iii) We have $\lim_{x \to +\infty} g(x) = \lim_{-\infty} g(x) = +\infty$.

(iv)
$$g'(x) = 12x^3 - 12x^2 - 72x = 12x(x^2 - x - 6) = 12x(x-3)(x+2).$$

This is defined everywhere, and has roots at $-2, 0, 3$.

Thus the critical points are $x = -2, 0, 3$. We compute $g(-2) = 0, g(0) = 64, g(3) = 5^2 \cdot 5 \cdot (-1) = -125$.

(v) We make a chart:

	$12x$	$(x-3)$	$(x+2)$	$g'(x)$
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

This implies relative minima at -2 and at 3, and a relative maximum at 0.

(vi) We compute
$$g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$$
which has roots
$$x = \frac{2 \pm \sqrt{4+72}}{6} = \frac{1 \pm \sqrt{19}}{3}.$$
and there are critical points at roughly -1 and $5/3$.

Plugging in some values we have

$$g''(2) = 12 \cdot (2) = 24 > 0$$
$$g''(0) = -72 < 0$$
$$g''(-2) = 12 \cdot 10 = 120 > 0.$$

Thus the function is concave up on $\left(-\infty, \frac{1-\sqrt{19}}{3}\right) \cup \left(\frac{1+\sqrt{19}}{3}, +\infty\right)$ and is concave down on $\left(\frac{1-\sqrt{19}}{3}, \frac{1+\sqrt{19}}{3}\right)$.

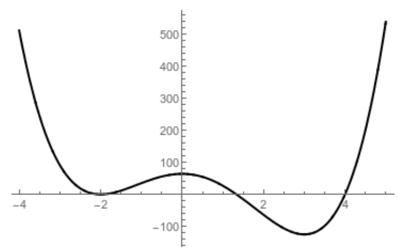

Figure 1: Graph of $g(x)$

Secondary Topic 8: Physical Optimization

Suppose that a company that produces and sells x units of a product each day makes a revenue of $R(x) = 260x - 9x^2/10$ dollars per day and has costs given by $C(x) = 1000 + 100x + x^2/10$ dollars per day. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$P'(x)160 - 2x$$
$$160 = 2x$$
$$80 = x$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

Thus our maximum profit is 5400 dollars per day.