Math 1231 Fall 2024
Single-Variable Calculus I Section 12
Mastery Quiz 11
Due Wednesday, November 13

This week's mastery quiz has two topics. Everyone should submit S9. If you have a 4/4 in M3, you don't need to submit it.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Secondary Topic 9: Riemann Sums

Name:

Recitation Section:

Major Topic 3: Optimization

(a) The function $f(x) = \dfrac{x^2+5}{x+2}$ has absolute extrema either on the interval $[-3,0]$ or on the interval $[0,3]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on the closed interval $[0,3]$, so it must have extrema there. (It is *not* continuous on $[-3,0]$ because it is undefined at -2.)

We compute

$$\begin{aligned} f'(x) &= \frac{2x(x+2)-(x^2+5)}{(x+2)^2} = \frac{x^2+4x-5}{(x+2)^2} \\ &= \frac{(x+5)(x-1)}{(x+2)^2} \end{aligned}$$

is zero for $x = 1, -5$ and is undefined for $x = -2$, so those are the critical points. The only one we have to care about is $x = 1$.

$$\begin{aligned} f(0) &= 5/2 \\ f(1) &= 2 \\ f(2) &= 9/4 \end{aligned}$$

so the absolute minimum is 2 at 1, and the absolute maximum is 5/2 at 0.

(b) Find and classify all the critical points of $f(x) = x^4 + 8x^3 + 10x^2 + 1$.

Solution: We compute

$$f'(x) = 4x^3 + 24x^2 + 20x = 4x(x^2+6x+5) = 4x(x+5)(x+1).$$

This is equal to zero when $x = 0, -1, -5$.

We can use the second derivative test:

$$\begin{aligned} f''(x) &= 12x^2 + 48x + 20 \\ f''(0) &= 20 > 0 \\ f''(-1) &= 16 < 0 \\ f''(-5) &= 300 - 240 + 20 = 80 > 0 \end{aligned}$$

so we see that f has local minima at $x = 0, x = -5$, and a local maximum at $x = -1$.

Alternatively, we could compute a chart

	$4x$	$x+5$	$x+1$	$f'(x)$
$x < -5$	$-$	$-$	$-$	$-$
$-5 < x < -1$	$-$	$+$	$-$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$-$
$0 < x$	$+$	$+$	$+$	$+$

Thus we conclude that f has local minima at $x = -5, 0$ and a local maximum at $x = -1$.

Secondary Topic 9: Riemann Sums

Using **only the definition of Riemann sum** and your knowledge of limits, compute the exact area under the curve $x^3 + 2x$ between $x = -1$ and $x = 2$.

Solution: We compute

$$
\begin{aligned}
R_n &= \sum_{k=1}^{n} \frac{3}{n} f\left(-1 + \frac{3k}{n}\right) = \frac{3}{n} \sum_{k=1}^{n} (-1 + 3i/n)^3 + 2(-1 + 3k/n) \\
&= \frac{3}{n} \sum_{k=1}^{n} 27k^3/n^3 - 27k^2/n^2 + 9k/n - 1 - 2 + 6k/n \\
&= \frac{3}{n} \sum_{k=1}^{n} -3 + 15k/n - 27k^2/n^2 + 27k^3/n^3 \\
&= \frac{-9}{n} \sum_{k=1}^{n} 1 + \frac{45}{n^2} \sum_{k=1}^{n} k + \frac{-81}{n^3} \sum_{k=1}^{n} k^2 + \frac{81}{n^4} \sum_{k=1}^{n} k^3 \\
&= \frac{-9}{n} \cdot n + \frac{45}{n^2} \cdot \frac{n(n+1)}{2} - \frac{81}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
\lim_{n \to +\infty} R_n &= \lim_{n \to +\infty} \frac{-9}{n} \cdot n + \frac{45}{n^2} \cdot \frac{n(n+1)}{2} - \frac{81}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= -9 + \frac{45}{2} - 27 + \frac{81}{4} = \frac{27}{4}.
\end{aligned}
$$