

Math 1231-13: Single-Variable Calculus 1  
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Recitation 2

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**Problem 1.** Let  $f(x) = 5x + 2$ . We want to use an  $\varepsilon - \delta$  argument to compute  $\lim_{x \rightarrow 2} f(x)$ .

- (a) If  $x$  is about 2, what should  $f(x)$  be?
- (b) Write down expressions using absolute value for the input and output errors.
- (c) If we want  $\varepsilon = 1$ , what does  $\delta$  need to be?
- (d) Find a formula for  $\delta$  in terms of  $\varepsilon$  (same form as  $\delta = \varepsilon/3$  or  $\delta = \varepsilon$ ).
- (e) Try to write a full proof.

**Solution:**

- (a)  $f(x) \approx 12$ .
- (b) Output error is  $|f(x) - 12|$  or  $|5x + 2 - 12|$ , which we can simplify to  $|5x - 10|$ . Input error is  $|x - 2|$ .
- (c) We want  $|5x - 10| < 1$ , and dividing by 5 gives  $|x - 2| < 1/5$ . So we'd need  $\delta = 1/5$ .
- (d) We want  $|5x - 10| < \varepsilon$ , and dividing by 5 gives  $|x - 2| < \varepsilon/5$ . So we'd need  $\delta = \varepsilon/5$ .
- (e) Let  $\varepsilon > 0$  and set  $\delta = \varepsilon/5$ . Then if  $0 < |x - 2| < \delta = \varepsilon/5$  we compute that

$$|f(x) - 12| = |5x - 10| = 5|x - 2| < 5 \cdot \varepsilon/5 = \varepsilon.$$

But we mostly want to practice the way we actually compute limits.

**Problem 2** (Warmup). Let  $f(x) = \frac{x^2 + \sin(x) + 3}{x^2 - x - 2}$ .

- (a) Where is  $f$  continuous? Where is it discontinuous?
- (b) What is  $\lim_{x \rightarrow 0} f(x)$ ?

**Solution:**

- (a) This function is made of algebra and trigonometry, so it's continuous where it's defined. The denominator is  $x^2 - x - 2 = (x - 2)(x + 1)$  so the function is undefined at 2 and  $-1$ .
- (b) Because this function is continuous at 0, we can just plug in:

$$\lim_{x \rightarrow 0} f(x) = f(0) = \frac{0^2 + \sin(0) + 3}{0^2 - 0 - 2} = -3/2.$$

**Problem 3.** Let  $f(x) = \frac{x-1}{x^2-1}$ .

- (a) What is  $f(2)$ ? Is  $f$  continuous at 2?
- (b) What is  $\lim_{x \rightarrow 2} f(x)$ ?
- (c) What is  $f(1)$ ? Is  $f$  continuous at 1?
- (d) What function can we find that's almost the same as  $f$ , but defined and continuous at 1? (Is this function the same as  $f$ ?)
- (e) What is  $\lim_{x \rightarrow 1} f(x)$ ?

**Solution:**

- (a)  $f(2) = 1/3$ , and  $f$  is continuous here since it's a reasonable functions.
- (b)  $\lim_{x \rightarrow 2} f(x) = 1/3$ .
- (c)  $f(1)$  isn't defined, and thus  $f$  is not continuous at 1.
- (d)  $\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$  is almost the same as  $\frac{1}{x+1}$ .
- (e)  $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$ .

Note we're using the Almost Identical Functions principle here. The function  $f$  was *not* continuous at 1, because it's not defined there. But we can replace it by an almost identical function that is continuous, and then that limit is simple to compute.

**Problem 4.** Let  $g(x) = \frac{(x+1)^2-1}{x+2}$ .

- (a) Is  $g$  continuous where it's defined? Where is it undefined?
- (b) Can you find a function that's almost identical to  $g$  but continuous everywhere?
- (c) What is  $\lim_{x \rightarrow -2} g(x)$ ?

**Solution:**

- (a)  $g$  is a reasonable function so it's continuous where it's defined, but it isn't defined at  $x = -2$ .
- (b)  $\frac{x^2+2x+1-1}{x+1} = \frac{x(x+2)}{x+2}$  is almost the same as  $x$ . So  $g(x)$  is almost the same as  $x$ .
- (c)  $\lim_{x \rightarrow -2} g(x) = \lim_{x \rightarrow -2} x = -2$ .

Note that  $\frac{x(x+2)}{x+2} \neq x$ , but their limits at 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).