Math 1231: Single-Variable Calculus 1
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Recitation 2

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Problem 1 (Warmup)**.** Let $f(x) = \frac{x^2+\sin(x)+3}{x^2-x-2}$.

(a) Where is f continuous? Where is it discontinuous?

(b) What is $\lim_{x\to 0} f(x)$?

Solution:

(a) This function is made of algebra and trigonometry, so it's continuous where it's defined. The denominator is $x^2 - x - 2 = (x-2)(x+1)$ so the function is undefined at 2 and -1.

(b) Because this function is continuous at 0, we can just plug in:

$$\lim_{x\to 0} f(x) = f(0) = \frac{0^2 + \sin(0) + 3}{0^2 - 0 - 2} = -3/2.$$

Problem 2. Let $f(x) = \frac{x-1}{x^2-1}$.

(a) What is $f(2)$? Is f continuous at 2?

(b) What is $\lim_{x\to 2} f(x)$?

(c) What is $f(1)$? Is f continuous at 1?

(d) What function can we find that's almost the same as f, but defined and continuous at 1? (Is this function the same as f?)

(e) What is $\lim_{x\to 1} f(x)$?

Solution:

(a) $f(2) = 1/3$, and f is continuous here since it's a reasonable functions.

(b) $\lim_{x\to 2} f(x) = 1/3$.

(c) $f(1)$ isn't defined, and thus f is not continuous at 1.

(d) $\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$ is almost the same as $\frac{1}{x+1}$.

(e) $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

Note we're using the Almost Identical Functions principle here. The function f was *not* continuous at 1, because it's not defined there. But we can replace it by an almost identical function that is continuous, and then that limit is simple to compute.

Problem 3. Let $g(x) = \frac{(x+1)^2-1}{x+2}$.

(a) Is g continuous where it's defined? Where is it undefined?

(b) Can you find a function that's almost identical to g but continuous everywhere?

(c) What is $\lim_{x\to -2} g(x)$?

Solution:

(a) g is a reasonable function so it's continuous where it's defined, but it isn't defined at $x = -2$.

(b) $\frac{x^2+2x+1-1}{x+1} = \frac{x(x+2)}{x+2}$ is almost the same as x. So $g(x)$ is almost the same as x.

(c) $\lim_{x\to -2} g(x) = \lim_{x\to -2} x = -2$.

Note that $\frac{x(x+2)}{x+2} \neq x$, but their limits at 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).

Problem 4. Let $h(x) = \frac{x-1}{\sqrt{5-x}-2}$.

(a) Is this function continuous where it's defined? Where is it undefined?

(b) We can factor an $x-1$ out of the top, but we can't obviously factor one out of the bottom. We need to use an algebraic trick make the $x-1$ appear. What tricks do we have that might work?

(c) What is $\lim_{X\to 1} h(x)$?

Solution:

(a) The function is reasonable, so it's continuous where defined. IT's undefined at $x = 1$ and also at $x > 5$.

(b) Here we need to multiply by the conjugate. We can compute

$$\begin{aligned}\frac{x-1}{\sqrt{5-x}-2} &= \frac{x-1}{\sqrt{5-x}-2}\frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}.\end{aligned}$$

This function is not the same as $-\sqrt{5-x}-2$, but it's very close.

(c)

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2}\frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\ &= \lim_{x\to 1}-(\sqrt{5-x}+2) = -4.\end{aligned}$$

Problem 5. We want to compute $\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}$.

(a) What rule do we know we need to invoke here?

(b) What θ are we going to need to pick for this to work out, and why?

(c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)

(d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

(a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.

(b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.

(c) We need to get a θ on the bottom as well. So we take

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} =^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

You have to use AIF in that first equality, because you're making the function undefined at -3.

There are a couple different ways to make this algebra work out. You might also try

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} =^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)/x^2-9 \cdot x^2-9}{x-3}$$
$$= \lim_{x\to 3} \frac{x^2-9}{x-3} =^{AIF} \lim_{x\to 3} x+3 = 6,$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} = \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$