Math 1231-13: Single-Variable Calculus 1
George Washington University Fall 2024
Recitation 3

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Friday September 13, 2024

Problem 1. Let $h(x) = \frac{x-1}{\sqrt{5-x}-2}$.

(a) Is this function continuous where it's defined? Where is it undefined?

(b) We can factor an $x-1$ out of the top, but we can't obviously factor one out of the bottom. We need to use an algebraic trick make the $x-1$ appear. What tricks do we have that might work?

(c) What is $\lim_{X \to 1} h(x)$?

Solution:

(a) The function is reasonable, so it's continuous where defined. IT's undefined at $x = 1$ and also at $x > 5$.

(b) Here we need to multiply by the conjugate. We can compute

$$\begin{aligned}
\frac{x-1}{\sqrt{5-x}-2} &= \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}.
\end{aligned}$$

This function is not the same as $-\sqrt{5-x}-2$, but it's very close.

(c)

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\
&= \lim_{x\to 1} -(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Problem 2. We want to compute $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$.

(a) What rule do we know we need to invoke here?

(b) What θ are we going to need to pick for this to work out, and why?

(c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)

(d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

(a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.

(b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.

(c) We need to get a θ on the bottom as well. So we take

$$\begin{aligned}
\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} &=^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9} \\
&= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.
\end{aligned}$$

You have to use AIF in that first equality, because you're making the function undefined at -3.

There are a couple different ways to make this algebra work out. You might also try

$$\begin{aligned}
\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} &=^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)/x^2-9 \cdot x^2-9}{x-3} \\
&= \lim_{x\to 3} \frac{x^2-9}{x-3} =^{AIF} \lim_{x\to 3} x+3 = 6,
\end{aligned}$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} &= \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9} \\ &= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.\end{aligned}$$

Problem 3. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x\to 0} 1/x$, and why?

(b) Let's look at $\lim_{x\to 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

(e) Compute $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?

(f) Now consider $\lim_{x\to 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?

(g) Finally, let's look at $\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?

(h) What pattern do you see from all of these?

Solution:

(a) $\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0}} = \pm\infty$.

(b) This looks like $\infty + \infty$ as an indeterminate form.

(c) We see $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x} = \lim_{x\to 0} \frac{2^{\nearrow 2}}{x_{\searrow 0}} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

(d) We already know $\lim_{x\to 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x\to 0} \frac{x-1^{\nearrow -1}}{x-x^2{}_{\searrow 0}} = \pm\infty.$$

So this is again $\infty + \infty$.

(e) By combining fractions, we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x\to 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x\to 0} 0 = 0.$$

So here $\infty + \infty = 0$.

(f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x+1^{\nearrow 1}}{x^2{}_{\searrow 0}} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

(g) We compute $\lim_{x\to 0} \frac{x-3x+2^{\nearrow 2}}{x^2-2x{}_{\searrow 0}} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x-2+x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x^2-2x}{x^2-2x} = \lim_{x\to 0} 1 = 1.$$

So here $\infty + \infty = 1$.

(h) In conclusion, if you know something looks like $\infty + \infty$, you don't really know anything about it at all.