Math 1231: Single-Variable Calculus 1
George Washington University Fall 2024
Recitation 2

Jay Daigle

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Problem 1. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x\to 0} 1/x$, and why?

(b) Let's look at $\lim_{x\to 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

(e) Compute $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?

(f) Now consider $\lim_{x\to 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?

(g) Finally, let's look at $\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?

(h) What pattern do you see from all of these?

Problem 2. (a) Consider $\lim_{x\to -\infty} \frac{x}{x+1}$. Can you come up with a heuristic guess about what this limit is?

(b) Can you carefully justify your guess from part (a).

(c) Now consider $\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+x}}$, and come up with a heuristic estimate for the limit.

(d) Carefully justify your guess from part (c).

(e) How would either of those calculations change if we take the limit to the other infinity?

Problem 3.

(a) We want to compute $\lim_{x\to+\infty} \sqrt{x^2+1} - x$. Can we just plug in here, or is this an indeterminate form? Why?

(b) When we have an indeterminate form, we generally want to write it as a big fraction, simplify, and factor. How can we do that here? We have to use a technique from last week to really get this to work.

(c) Once you have a big fraction, use it to compute the limit.

(d) How does this argument change if instead we want $\lim_{x\to+\infty} \sqrt{x^2+x+1} - x$?

(e) What is $\lim_{x\to+\infty} \sqrt{x^2+ax+1} - x$?

(f) What does the answer in part (e) say about $\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x$? Why should the answer to this question be obvious?

Problem 4. Let $f(x) = x^3$. We want to find a formula for the derivative of this function at any given point.

(a) Write down a formula for $f'(a)$ using the $h \to 0$ limit formulation. What does the numerator mean? What does the denominator mean?

(b) Use your formula from part (a) to compute the derivative.

(c) Now write down a formula for $f'(a)$ using the $x \to a$ limit formulation. Does this look easier or harder than the formula from part (a), and why? What does the numerator mean? What does the denominator mean?

(d) Use the formula from part (c) to compute the derivative. You should get the same answer you got in part (b).

(e) Which method was faster? Which method was easier?

Problem 5. Let $g(x) = \sqrt[3]{x}$.

(a) Write down a limit formula to compute the derivative of g at 0.

(b) What is $g'(0)$? What does this tell you?

(c) Now write down a limit formula to compute the derivative of $p(x) = \sqrt[3]{x^2}$.

(d) What is this limit? What does that tell you?

(e) Write down a limit formula to compute the derivative of g at a when $a \neq 0$.

(f) (Bonus) Can you compute this limit? What do you have to do here? (It's not obvious, but there's an algebraic trick we've mentioned that can help us.)