Math 1231: Single-Variable Calculus 1
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Recitation 2

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Problem 1. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x\to 0} 1/x$, and why?

(b) Let's look at $\lim_{x\to 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

(e) Compute $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?

(f) Now consider $\lim_{x\to 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?

(g) Finally, let's look at $\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?

(h) What pattern do you see from all of these?

Solution:

(a) $\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0}} = \pm\infty$.

(b) This looks like $\infty + \infty$ as an indeterminate form.

(c) We see $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x} = \lim_{x\to 0} \frac{2^{\nearrow 2}}{x_{\searrow 0}} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

(d) We already know $\lim_{x\to 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x\to 0} \frac{x - 1^{\nearrow -1}}{x - x^2{}_{\searrow 0}} = \pm\infty.$$

So this is again $\infty + \infty$.

(e) By combining fractions, we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x\to 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x\to 0} 0 = 0.$$

So here $\infty + \infty = 0$.

(f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x + 1^{\nearrow 1}}{x^2{}_{\searrow 0}} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

(g) We compute $\lim_{x\to 0} \frac{x-3x+2^{\nearrow 2}}{x^2-2x_{\searrow 0}} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x-2+x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x^2-2x}{x^2-2x} = \lim_{x\to 0} 1 = 1.$$

So here $\infty + \infty = 1$.

(h) In conclusion, if you know something looks like $\infty+\infty$, you don't really know anything about it at all.

Problem 2. (a) Consider $\lim_{x\to -\infty} \frac{x}{x+1}$. Can you come up with a heuristic guess about what this limit is?

(b) Can you carefully justify your guess from part (a).

(c) Now consider $\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+x}}$, and come up with a heuristic estimate for the limit.

(d) Carefully justify your guess from part (c).

(e) How would either of those calculations change if we take the limit to the other infinity?

Solution:

(a) If x is large, x and $x+1$ should behave basically the same; the 1 is insignificant compared to the x. So this limit should be 1.

(b)
$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$

(c) When x is large, the x will be really large, but really small relative to the $3x^2$. So this should look like $\frac{x}{\sqrt{3x^2}}$ which goes to $\frac{1}{\sqrt{3}}$.

(d)
$$\begin{aligned}
\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/x} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/\sqrt{x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{3+\frac{1}{x^2}}} = \frac{1}{\sqrt{3}}.
\end{aligned}$$

(e) The first wouldn't change at all. The second would change, because if $x > 0$ then $x = \sqrt{x^2}$, but if $x < 0$ then $x = -\sqrt{x^2}$. So we instead get
$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/x} \\
&= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/-\sqrt{x^2}} \\
&= \lim_{x\to-\infty} \frac{-1}{\sqrt{3+\frac{1}{x^2}}} = \frac{-1}{\sqrt{3}}.
\end{aligned}$$

Problem 3.

(a) We want to compute $\lim_{x\to+\infty} \sqrt{x^2+1} - x$. Can we just plug in here, or is this an indeterminate form? Why?

(b) When we have an indeterminate form, we generally want to write it as a big fraction, simplify, and factor. How can we do that here? We have to use a technique from last week to really get this to work.

(c) Once you have a big fraction, use it to compute the limit.

(d) How does this argument change if instead we want $\lim_{x\to+\infty}\sqrt{x^2+x+1}-x$?

(e) What is $\lim_{x\to+\infty}\sqrt{x^2+ax+1}-x$?

(f) What does the answer in part (e) say about $\lim_{x\to+\infty}\sqrt{x^2+2x+1}-x$? Why should the answer to this question be obvious?

Solution:

(a) This is indeterminate, of the form $\infty-\infty$.

(b) The simplest thing we could do is just write

$$\lim_{x\to+\infty}\sqrt{x^2+1}-x=\lim_{x\to+\infty}\frac{\sqrt{x^2+1}-x}{1}.$$

But that doesn't get us very far. Since we have a difference of square roots, we want to multiply by the conjugate.

(c) We get

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+1}-x&=\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\&=\lim_{x\to+\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}\\&=\lim_{x\to+\infty}\frac{1/x}{\sqrt{1+1/x^2}+1}=0.\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+x+1}-x&=\lim_{x\to+\infty}\left(\sqrt{x^2+x+1}-x\right)\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}\\&=\lim_{x\to+\infty}\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}=\lim_{x\to+\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}\\&=\lim_{x\to+\infty}\frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1}=\frac{1}{2}.\end{aligned}$$

(e) We can make essentially the same argument:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{x^2+ax+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+ax+1} - x\right) \frac{\sqrt{x^2+ax+1}+x}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{x^2+ax+1-x^2}{\sqrt{x^2+ax+1}+x} = \lim_{x\to+\infty} \frac{ax+1}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{a+1/x}{\sqrt{1+a/x+1/x^2}+1} = \frac{a}{2}.
\end{aligned}$$

(f) By the answer from part (b), $\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x = 2/2 = 1$. But we could also just observe that $x^2+2x+1 = (x+1)^2$, so

$$\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x = \lim_{x\to+\infty} (x+1) - x = 1.$$

Problem 4. Let $f(x) = x^3$. We want to find a formula for the derivative of this function at any given point.

(a) Write down a formula for $f'(a)$ using the $h \to 0$ limit formulation. What does the numerator mean? What does the denominator mean?

(b) Use your formula from part (a) to compute the derivative.

(c) Now write down a formula for $f'(a)$ using the $x \to a$ limit formulation. Does this look easier or harder than the formula from part (a), and why? What does the numerator mean? What does the denominator mean?

(d) Use the formula from part (c) to compute the derivative. You should get the same answer you got in part (b).

(e) Which method was faster? Which method was easier?

Solution:

(a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h}$$

The top is the difference between two output values; the bottom is the difference between the corresponding inputs. You can think of the bottom as "change the input by a bit" and the top as the difference between the two outputs.

(b)

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &=^{AIF} \lim_{h \to 0} 3a^2 + 3ah + h^2 = 3a^2. \end{aligned}$$

(c)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^3 - a^3}{x - a}$$

Again the top is the difference in outputs and the bottom is the difference in inputs. Here we can see two specific inputs, x and a, on the bottom; the top is the two corresponding outputs.

(d)

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} \\ &=^{AIF} \lim_{x \to a} x^2 + ax + a^2 = 3a^2. \end{aligned}$$

Notice we use the difference of cubes formula from section 1.1 of the notes.

(e) To my eyes, at least, the $h \to 0$ method is more straightforward, but the $x \to a$ method is faster *if you know the trick*. If you look at it and immediately see that $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$, then the $x \to a$ method works very quickly. But if you don't know or remember that fact, it's hard to figure out what to do at all; you just get stuck.

In contrast, the $h \to 0$ method takes more algebra and work and writing and time, but less cleverness and thinking. If you just multiply everything out and cancel out the obvious stuff, it works out. When I don't know what I'm doing, I default to the $h \to 0$ version.

Problem 5. Let $g(x) = \sqrt[3]{x}$.

(a) Write down a limit formula to compute the derivative of g at 0.

(b) What is $g'(0)$? What does this tell you?

(c) Now write down a limit formula to compute the derivative of $p(x) = \sqrt[3]{x^2}$.

(d) What is this limit? What does that tell you?

(e) Write down a limit formula to compute the derivative of g at a when $a \neq 0$.

(f) (Bonus) Can you compute this limit? What do you have to do here? (It's not obvious, but there's an algebraic trick we've mentioned that can help us.)

Solution:

(a)

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h} \\ &= \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x} - 0}{x - 0}. \end{aligned}$$

(b)

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

This is a vertical tangent line, because the limit is always $+\infty$.

(c)

$$\begin{aligned} p'(0) &= \lim_{h\to 0} \frac{p(h) - p(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h^2} - 0}{h} \\ &= \lim_{x \to 0} \frac{p(x) - p(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x^2} - 0}{x - 0}. \end{aligned}$$

(d)

$$p'(0) = \lim_{h\to 0} \frac{p(h) - p(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h^2}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

This is a *cusp*, because the limit is $\pm\infty$ rather than just $+\infty$.

(e)

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(h) - g(a)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h} \\ &= \lim_{x \to 0} \frac{g(x) - g(a)}{x - a} = \lim_{x\to 0} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}. \end{aligned}$$

(f) You might recognize this as being a difference of cube roots, so we can use the difference-of-cubes formula, as a sort of generalization of multiplication by the conjugate.

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{\sqrt[3]{a+h} - \sqrt[3]{a}}{h} \\
&= \lim_{h\to 0} \frac{(a+h) - a}{h(\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt[3]{(a+h)^2} + \sqrt[3]{(a+h)a} + \sqrt[3]{a^2}} \\
&= \frac{1}{\sqrt[3]{a^2} + \sqrt[3]{a^2} + \sqrt[3]{a^2}} = \frac{1}{3\sqrt[3]{a^2}}.
\end{aligned}$$