

Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 4
Due Monday, September 23

This week's mastery quiz has three topics. Everyone should submit on M2 and S2. If you have a 4/4 on M1 in Blackboard, you don't need to submit it again; but if you have less than 4/4, you should give it another go.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 2: Definition of Derivative

Name:

Recitation Section:

Major Topic 1: Computing Limits

$$(a) \lim_{x \rightarrow 3} \frac{1-x}{(x-3)^3} =$$

Solution: The top approaches -2 and the bottom approaches 0, so

$$\lim_{x \rightarrow 3} \frac{1-x}{(x-3)^3} = \pm\infty.$$

Since the denominator can be either positive or negative, we can't be any more precise than this.

$$(b) \lim_{x \rightarrow +\infty} \frac{3x^2 + 2x + 1}{\sqrt{x^4 - x^2 + x}} =$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{3x^2 + 2x + 1}{\sqrt{x^4 - x^2 + x}} &= \lim_{x \rightarrow +\infty} \frac{3 + 2/x + 1/x^2}{1/\sqrt{x^4} \sqrt{x^4 - x^2 + x}} \\ &= \lim_{x \rightarrow +\infty} \frac{3 + 2/x + 1/x^2}{\sqrt{1 - 1/x^2 + 1/x^3}} \\ &= \frac{3 + 0 + 0}{\sqrt{1 - 0 + 0}} = 3. \end{aligned}$$

$$(c) \lim_{x \rightarrow 1} \frac{\sqrt{x+3} - 2}{x-1} =$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x+3} - 2}{x-1} &= \lim_{x \rightarrow 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{4}. \end{aligned}$$

Major Topic 2: Computing Derivatives

- (a) Compute the derivative of $f(x) = (2x^4 + 3)(5x - 2\sqrt{x})$, explicitly justifying each step and naming each derivative rule you use.

Solution:

$$\begin{aligned}
f'(x) &= (2x^4 + 3)'(5x - 2\sqrt{x}) + (2x^4 + 3)(5x - 2\sqrt{x})' && \text{Product Rule} \\
&= \left((2x^4)' + (3)' \right) (5x - 2\sqrt{x}) + (2x^4 + 3) \left((5x)' - (2\sqrt{x})' \right) && \text{Sum Rule} \\
&= \left((2x^4) + 0 \right) (5x - 2\sqrt{x}) + (2x^4 + 3) \left((5x)' - (2\sqrt{x})' \right) && \text{Constants Rule} \\
&= \left(2(x^4)' \right) (5x - 2\sqrt{x}) + (2x^4 + 3) \left(5(x)' - 2(\sqrt{x})' \right) && \text{Scalar Products} \\
&= \left(2(x^4)' \right) (5x - 2\sqrt{x}) + (2x^4 + 3) \left(5 \cdot 1 - 2(\sqrt{x})' \right) && \text{Identity} \\
&= (2 \cdot 4x^3)(5x - 2\sqrt{x}) + (2x^4 + 3) \left(5 \cdot 1 - 2 \cdot \frac{1}{2} x^{-1/2} \right) && \text{Power Rule.}
\end{aligned}$$

(b) Compute the derivative of $h(x) = \frac{5}{x^4}$.

Solution: There are two ways you could approach this but you need to do it correctly consistently; this is question that trips people up very frequently.

One option is to use the quotient rule. But if we do that we have to remember that $5' = 0$. So we get

$$h'(x) = \frac{(5)'x^4 - 5(x^4)'}{(x^4)^2} = \frac{0 - 20x^3}{x^8} = \frac{-20}{x^5}.$$

The other option is to rewrite this as an exponent. Then we have

$$h'(x) = \frac{d}{dx} 5x^{-4} = 5 \cdot (-4x^{-5}) = -20x^{-5}.$$

I think the second approach is better and more consistent. But you *can* use the quotient rule as long as you do it correctly.

(c) $\frac{d}{dx} (5x^7 - 3x) \left(x^{4/3} + \frac{1}{x} \right) =$

Solution:

$$(35x^6 - 3) \left(x^{4/3} + \frac{1}{x} \right) + \left(\frac{4}{3}x^{1/3} - \frac{1}{x^2} \right) (5x^7 - 3x).$$

Secondary Topic 2: Definition of Derivative

(a) If $f(x) = x^2 + 2x$, find $f'(2)$, explicitly using the definition of the derivative.

Solution:

$$\begin{aligned}f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\&= \lim_{h \rightarrow 0} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\&= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\&= \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} \\&= \lim_{h \rightarrow 0} h + 6 = 6.\end{aligned}$$

(b) If $g(x) = \frac{x}{x+2}$, find $g'(x)$ explicitly using the definition of derivative.

Solution:

$$\begin{aligned}g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} \\&= \lim_{h \rightarrow 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x+2)(x+h+2)} \\&= \lim_{h \rightarrow 0} \frac{x^2 + hx + 2x + 2h - x^2 - xh - 2x}{h(x+2)(x+h+2)} \\&= \lim_{h \rightarrow 0} \frac{2h}{h(x+2)(x+h+2)} \\&= \lim_{h \rightarrow 0} \frac{2}{(x+2)(x+h+2)} = \frac{2}{(x+2)^2}.\end{aligned}$$