Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 4
Due Wednesday, September 25

This week's mastery quiz has three topics. Everyone should submit M2. If you have a 4/4 on M1 (meaning you've gotten two different 2s), you don't need to submit it. If you have a 2/2 on S2, you don't need to submit it. If you're unsure about your current grade, please check Blackboard.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 2: Definition of Derivative

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim\limits_{x\to 0} \dfrac{\sin(5x^2)+\tan^2(x)}{x^2} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(5x^2)+\tan^2(x)}{x^2} &= \lim_{x\to 0} 5\frac{\sin(5x^2)}{5x^2} + \lim_{x\to 0}\left(\frac{\sin(x)}{\cos(x)\cdot x}\right)^2 \\ &= 5+1^2 = 6\end{aligned}$$

by the Small Angle Approximation.

(b) Compute $\lim\limits_{x\to -1} \dfrac{1-x}{1+x} =$

Solution: The limit of the top is 2 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the denominator can be positive or negative, we can't be more specific.

(c) Compute $\lim\limits_{x\to 2} \dfrac{4}{x-2} - \dfrac{6x}{x^2-x-2} =$

Solution:

$$\begin{aligned}\lim_{x\to 2} \frac{4}{x-2} - \frac{6x}{x^2-x-2} &= \lim_{x\to 2} \frac{4(x+1)}{(x+1)(x-2)} - \frac{6x}{(x+1)(x-2)} \\ &= \lim_{x\to 2} \frac{-2x+4}{(x+1)(x-2)} = \lim_{x\to 2} \frac{-2(x-2)}{(x+1)(x-2)} \\ &= \lim_{x\to 2} \frac{-2}{x+1} = \frac{-2}{3}.\end{aligned}$$

Major Topic 2: Computing Derivatives

(a) Explicitly justifying each step and naming each derivative rule you use, compute $\frac{d}{dx}\frac{\sin(x)+1}{2x^2-5}$.

Solution:

$$\begin{aligned}
\frac{d}{dx}\frac{\sin(x)+1}{2x^2-5} &= \frac{(\sin(x)+1)'(2x^2-5)-(2x^2-5)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Quotient rule}\\
&= \frac{\big((\sin(x))'+(1)'\big)(2x^2-5)-\big((2x^2)'-(5)'\big)(\sin(x)+1)}{(2x^2-5)^2} && \text{Sum Rule}\\
&= \frac{(\sin(x))'(2x^2-5)-(2x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Constants rule}\\
&= \frac{\cos(x)(2x^2-5)-(2x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Derivative of sine}\\
&= \frac{\cos(x)(2x^2-5)-2(x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Scalar Products}\\
&= \frac{\cos(x)(2x^2-5)-2(2x)(\sin(x)+1)}{(2x^2-5)^2} && \text{Power Rule.}
\end{aligned}$$

(b) Compute the derivative of $g(x) = \dfrac{5x^4 - 3x^2}{x^5 + \sqrt[5]{x} + 7}$.

Solution:

$$g'(x) = \frac{(20x^3 - 6x)(x^5 + \sqrt[5]{x} + 7) - (5x^4 + \frac{1}{5}x^{-4/5})(5x^4 - 3x^2)}{(x^5 + x + 7)^2}.$$

(c) Compute $\dfrac{d}{dx}\tan^{3/5}\big(\sec\big(x^3-4\big)\big)$

Solution:

$$\begin{aligned}
&\frac{3}{5}\tan^{-2/5}\big(\sec\big(x^3-4\big)\big)\sec^2\big(\sec\big(x^3-4\big)\big)\\
&\qquad\cdot\sec\big(x^3-4\big)\tan\big(x^3-4\big)\,3x^2
\end{aligned}$$

Secondary Topic 2: Definition of Derivative

(a) If $f(x) = 3x^2 - 4x$, find $f'(-3)$, explicitly using the definition of derivative.

Solution:

$$\begin{aligned} f'(-3) &= \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h} \\ &= \lim_{h\to 0} \frac{3(h-3)^2-4(h-3)-39}{h} \\ &= \lim_{h\to 0} \frac{3h^2-18h+27-4h+12-39}{h} \\ &= \lim_{h\to 0} \frac{3h^2-22h}{h} \\ &= \lim_{h\to 0} 3h-22 = -22. \end{aligned}$$

(b) If $g(x) = \sqrt{x-5}$, find $g'(x)$, directly from the formal definition of the derivative.

Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h-5}-\sqrt{x-5}}{h} \\ &= \lim_{h\to 0} \frac{x+h-5-(x-5)}{h(\sqrt{x+h-5}+\sqrt{x-5})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-5}+\sqrt{x-5})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h-5}+\sqrt{x-5}} = \frac{1}{2\sqrt{x-5}}. \end{aligned}$$