Math 1231-13: Single-Variable Calculus 1
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Recitation 4

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Problem 1. (a) Consider $\lim_{x\to-\infty} \frac{x}{x+1}$. Can you come up with a heuristic guess about what this limit is?

(b) Can you carefully justify your guess from part (a).

(c) Now consider $\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+x}}$, and come up with a heuristic estimate for the limit.

(d) Carefully justify your guess from part (c).

(e) How would either of those calculations change if we take the limit to the other infinity?

Solution:

(a) If x is large, x and $x+1$ should behave basically the same; the 1 is insignificant compared to the x. So this limit should be 1.

(b)
$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$

(c) When x is large, the x will be really large, but really small relative to the $3x^2$. So this should look like $\frac{x}{\sqrt{3x^2}}$ which goes to $\frac{1}{\sqrt{3}}$.

(d)

$$\begin{aligned}
\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/x} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/\sqrt{x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{3+\frac{1}{x^2}}} = \frac{1}{\sqrt{3}}.
\end{aligned}$$

(e) The first wouldn't change at all. The second would change, because if $x > 0$ then $x = \sqrt{x^2}$, but if $x < 0$ then $x = -\sqrt{x^2}$. So we instead get

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/x} \\
&= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/-\sqrt{x^2}} \\
&= \lim_{x\to-\infty} \frac{-1}{\sqrt{3+\frac{1}{x^2}}} = \frac{-1}{\sqrt{3}}.
\end{aligned}$$

Problem 2.

(a) We want to compute $\lim_{x\to+\infty} \sqrt{x^2+x+1} - x$?

(b) What is $\lim_{x\to+\infty} \sqrt{x^2+ax+1} - x$?

(c) What does the answer in part (b) say about $\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x$? Why should the answer to this question be obvious?

Solution:

(a)

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{x^2+x+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+x+1} - x\right) \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} \\
&= \lim_{x\to+\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} \\
&= \lim_{x\to+\infty} \frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1} = \frac{1}{2}.
\end{aligned}$$

(b) We can make essentially the same argument:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{x^2+ax+1}-x &= \lim_{x\to+\infty} \left(\sqrt{x^2+ax+1}-x\right)\frac{\sqrt{x^2+ax+1}+x}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{x^2+ax+1-x^2}{\sqrt{x^2+ax+1}+x} = \lim_{x\to+\infty} \frac{ax+1}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{a+1/x}{\sqrt{1+a/x+1/x^2}+1} = \frac{a}{2}.
\end{aligned}$$

(c) By the answer from part (b), $\lim_{x\to+\infty} \sqrt{x^2+2x+1}-x = 2/2 = 1$. But we could also just observe that $x^2+2x+1 = (x+1)^2$, so

$$\lim_{x\to+\infty} \sqrt{x^2+2x+1}-x = \lim_{x\to+\infty} (x+1)-x = 1.$$

Problem 3. Let $f(x) = x^3$. We want to find a formula for the derivative of this function at any given point.

(a) Write down a formula for $f'(a)$ using the $h \to 0$ limit formulation. What does the numerator mean? What does the denominator mean?

(b) Use your formula from part (a) to compute the derivative.

(c) Now write down a formula for $f'(a)$ using the $x \to a$ limit formulation. Does this look easier or harder than the formula from part (a), and why? What does the numerator mean? What does the denominator mean?

(d) Use the formula from part (c) to compute the derivative. You should get the same answer you got in part (b).

(e) Which method was faster? Which method was easier?

Solution:

(a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^3-a^3}{h}$$

The top is the difference between two output values; the bottom is the difference between the corresponding inputs. You can think of the bottom as “change the input by a bit” and the top as the difference between the two outputs.

(b)

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\ &=^{AIF} \lim_{h\to 0} 3a^2 + 3ah + h^2 = 3a^2. \end{aligned}$$

(c)

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^3 - a^3}{x - a}$$

Again the top is the difference in outputs and the bottom is the difference in inputs. Here we can see two specific inputs, x and a, on the bottom; the top is the two corresponding outputs.

(d)

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} \\ &=^{AIF} \lim_{x\to a} x^2 + ax + a^2 = 3a^2. \end{aligned}$$

Notice we use the difference of cubes formula from section 1.1 of the notes.

(e) To my eyes, at least, the $h \to 0$ method is more straightforward, but the $x \to a$ method is faster *if you know the trick*. If you look at it and immediately see that $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$, then the $x \to a$ method works very quickly. But if you don't know or remember that fact, it's hard to figure out what to do at all; you just get stuck.

In contrast, the $h \to 0$ method takes more algebra and work and writing and time, but less cleverness and thinking. If you just multiply everything out and cancel out the obvious stuff, it works out. When I don't know what I'm doing, I default to the $h \to 0$ version.

Problem 4. Let $a(x) = |x|$ be the absolute value function.

(a) Write down a formula for a as a piecewise function.

(b) Write down a limit expression for the derivative of a at 0.

(c) What is the limit from the right?

(d) What is the limit from the left?

(e) What does that tell you about the derivative?

Solution:

(a)

$$a(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

(b)

$$a'(0) = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

(c)

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

(d)

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

(e) The limits to the right and the left don't exist, so the limit doesn't exist.

Problem 5. Let $g(x) = \sqrt[3]{x}$.

(a) Write down a limit formula to compute the derivative of g at 0.

(b) What is $g'(0)$? What does this tell you?

(c) Now write down a limit formula to compute the derivative of $p(x) = \sqrt[3]{x^2}$.

(d) What is this limit? What does that tell you?

(e) Write down a limit formula to compute the derivative of g at a when $a \neq 0$.

(f) (Bonus) Can you compute this limit? What do you have to do here? (It's not obvious, but there's an algebraic trick from Day 1 that can help us.)

Solution:

(a)

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}-0}{h} \\ &= \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{\sqrt[3]{x}-0}{x-0}. \end{aligned}$$

(b)

$$g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

This is a vertical tangent line, because the limit is always $+\infty$.

(c)

$$\begin{aligned} p'(0) &= \lim_{h\to 0} \frac{p(h)-p(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h^2}-0}{h} \\ &= \lim_{x\to 0} \frac{p(x)-p(0)}{x-0} = \lim_{x\to 0} \frac{\sqrt[3]{x^2}-0}{x-0}. \end{aligned}$$

(d)

$$p'(0) = \lim_{h\to 0} \frac{p(h)-p(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h^2}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

This is a *cusp*, because the limit is $\pm\infty$ rather than just $+\infty$.

(e)

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(h)-g(a)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{a+h}-\sqrt[3]{a}}{h} \\ &= \lim_{x\to 0} \frac{g(x)-g(a)}{x-a} = \lim_{x\to 0} \frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}. \end{aligned}$$

(f) You might recognize this as being a difference of cube roots, so we can use the difference-of-cubes formula, as a sort of generalization of multiplication by the conjugate.

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{\sqrt[3]{a+h}-\sqrt[3]{a}}{h} \\ &= \lim_{h\to 0} \frac{(a+h)-a}{h(\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt[3]{(a+h)^2}+\sqrt[3]{(a+h)a}+\sqrt[3]{a^2}} \\ &= \frac{1}{\sqrt[3]{a^2}+\sqrt[3]{a^2}+\sqrt[3]{a^2}} = \frac{1}{3\sqrt[3]{a^2}}. \end{aligned}$$

Problem 6. Let $f(x) = \sqrt{x^2 - 4}$.

(a) Set up a limit expression to calculate $f'(x)$. Do you think $h \to 0$ or $x \to a$ will be easier here?

(b) Compute $f'(x)$.

(c) Where is f differentiable? Where is it not differentiable?

Solution:

(a) We could say

$$f'(x) = \lim_{x \to a} \frac{\sqrt{x^2 - 4} - \sqrt{a^2 - 4}}{x - a}$$
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 4} - \sqrt{x^2 - 4}}{h}.$$

The first would in fact work, but the second looks easier in every way to me.

(b)

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 4} - \sqrt{x^2 - 4}}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - 4 - (x^2 - 4)}{h(\sqrt{(x+h)^2 - 4} + \sqrt{x^2 - 4})} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2 - 4} + \sqrt{x^2 - 4})} \\ &= \lim_{h \to 0} \frac{2x + h}{(\sqrt{(x+h)^2 - 4} + \sqrt{x^2 - 4})} \\ &= \frac{2x}{2\sqrt{x^2 + 4}} = \frac{x}{\sqrt{x^2 - 4}}. \end{aligned}$$

(c) This derivative is defined for $x < -2$ and for $x > 2$, but not in between those two numbers. Thus we see that f is differentiable on $(-\infty, -2) \cup (2, +\infty)$.

Problem 7. Let $g(x) = \frac{1}{x+3}$.

(a) Write down a limit expression to compute $g'(2)$. Be careful with order of operations and parentheses!

(b) Now compute $g'(2)$.

(c) Write a limit expression to compute $g'(x)$. Again, make sure you get your order of operations right.

(d) Compute $g'(x)$.

Solution:

(a) We have

$$g'(2) = \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}.$$

Make sure you have $\frac{1}{5+h}$, and not $\frac{1}{5} + h$! The second thing is very different and will not give you a useful answer.

(b) We have

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{5+h} - \frac{1}{5}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{5-(5+h)}{5(5+h)}\right) \\ &= \lim_{h\to 0} \frac{-h}{5h(5+h)} = \lim_{h\to 0} \frac{-1}{5(5+h)} \\ &= \frac{-1}{5(5+0)} = \frac{-1}{25}. \end{aligned}$$

(c)

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}.$$

Again, we want to make sure that we don't write $\frac{1}{x+3} + h$ or something like that.

(d)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)}\right) \\ &= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\ &= \frac{-1}{(x+3)^2}. \end{aligned}$$