Math 1231: Single-Variable Calculus 1
George Washington University Fall 2024
Recitation 4

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Problem 1. Let $g(x) = \frac{1}{x+3}$.

(a) Write down a limit expression to compute $g'(2)$. Be careful with order of operations and parentheses!

(b) Now compute $g'(2)$.

(c) Write a limit expression to compute $g'(x)$. Again, make sure you get your order of operations right.

(d) Compute $g'(x)$.

Solution:

(a) We have

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}. \end{aligned}$$

Make sure you have $\frac{1}{5+h}$, and not $\frac{1}{5} + h$! The second thing is very different and will not give you a useful answer.

(b) We have

$$\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{5+h} - \frac{1}{5}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{5-(5+h)}{5(5+h)}\right) \\
&= \lim_{h\to 0} \frac{-h}{5h(5+h)} = \lim_{h\to 0} \frac{-1}{5(5+h)} \\
&= \frac{-1}{5(5+0)} = \frac{-1}{25}.
\end{aligned}$$

(c)

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}.$$

Again, we want to make sure that we don't write $\frac{1}{x+3} + h$ or something like that.

(d)

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)}\right) \\
&= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} \\
&= \frac{-1}{(x+3)^2}.
\end{aligned}$$

Problem 2. Let $a(x) = |x|$ be the absolute value function.

(a) Write down a formula for a as a piecewise function.

(b) Write down a limit expression for the derivative of a at 0.

(c) What is the limit from the right?

(d) What is the limit from the left?

(e) What does that tell you about the derivative?

Solution:

(a)
$$a(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

(b)
$$a'(0) = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

(c)
$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

(d)
$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$$

(e) The limits to the right and the left don't exist, so the limit doesn't exist.

Problem 3. Let $f(x) = \sqrt{x^2 - 4}$.

(a) Set up a limit expression to calculate $f'(x)$. Do you think $h \to 0$ or $x \to a$ will be easier here?

(b) Compute $f'(x)$.

(c) Where is f differentiable? Where is it not differentiable?

Solution:

(a) We could say

$$f'(x) = \lim_{x \to a} \frac{\sqrt{x^2 - 4} - \sqrt{a^2 - 4}}{x - a}$$
$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - 4} - \sqrt{x^2 - 4}}{h}.$$

The first would in fact work, but the second looks easier in every way to me.

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-4}-\sqrt{x^2-4}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2-4-(x^2-4)}{h(\sqrt{(x+h)^2-4}+\sqrt{x^2-4})} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2-4}+\sqrt{x^2-4})} \\ &= \lim_{h\to 0} \frac{2x+h}{(\sqrt{(x+h)^2-4}+\sqrt{x^2-4})} \\ &= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2-4}}. \end{aligned}$$

(c) This derivative is defined for $x < -2$ and for $x > 2$, but not in between those two numbers. Thus we see that f is differentiable on $(-\infty, -2) \cup (2, +\infty)$.

Problem 4. (a) Let $h(x) = \tan^2(x)$. Find functions f and g so that $h(x) = (f \circ g)(x)$.

(b) Compute $f'(x)$ and $g'(x)$. Use that info to compute $h'(x)$.

(c) Now let $h(x) = \tan(x^2)$. Find functions f and g so that $h(x) = (f \circ g)(x)$.

(d) Compute $f'(x)$ and $g'(x)$. Use that information to compute $h'(x)$.

Solution:

(a) We can take $f(x) = x^2$ and $g(x) = \tan(x)$.

(b) $f'(x) = 2x$ and $g'(x) = \sec^2(x)$, so

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(\tan(x)) \cdot g'(x) = 2\tan(x) \cdot \sec^2(x).$$

(c) Now we have $f(x) = \tan(x)$ and $g(x) = x^2$.

(d) Now we have $f'(x) = \sec^2(x)$ and $g'(x) = 2x$, so

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(x^2) \cdot g'(x) = \sec^2(x^2) \cdot 2x.$$

Problem 5. Consider the function $\sec^2(x^2+1)$

(a) Find functions f and g so that $(f \circ g)(x) = \sec^2(x^2+1)$.

(b) Talk to the people next to you. Did they pick the same f and g that you did? Can you find a different pair of functions f and g that also work?

(c) Find functions f, g, h so that $(f \circ g \circ h)(x) = \sec^2(x^2+1)$.

(d) Compute f', g', and h'.

(e) What is $\frac{d}{dx} \sec^2(x^2+1)$?

Solution:

(a) There are basically two choices here. You could say that $f(x) = \sec^2(x)$ and $g(x) = x^2+1$, which is maybe the more obvious choice; or you could say that $f(x) = x^2$ and $g(x) = \sec(x^2+1)$.

(b) This is really a composite of three functions, which is why you could make different choices here.

(c) $f(x) = x^2, g(x) = \sec(x), h(x) = x^2+1$. (Technically there are other things you could do, like $g(x) = \sec(x+1)$ and $h(x) = x^2$, but those are moderately silly.)

(d) $f'(x) = 2x, g'(x) = \sec(x)\tan(x), h'(x) = 2x$.

(e)

$$\begin{aligned}\frac{d}{dx}\sec^2\left(x^2+1\right) &= f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x) \\ &= f'(\sec(x^2+1)) \cdot g'(x^2+1) \cdot h'(x) \\ &= 2\sec(x^2+1) \cdot \sec(x^2+1)\tan(x^2+1) \cdot 2x.\end{aligned}$$

Problem 6 (Bonus)**.** Find

$$\frac{d}{dx} \frac{\sin(x^2) + \sin^2(x)}{x^2+1}$$

Solution:

$$\begin{aligned}\frac{d}{dx} \frac{\sin(x^2) + \sin^2(x)}{x^2+1} &= \frac{(\sin(x^2) + \sin^2(x))'(x^2+1) - 2x(\sin(x^2) + \sin^2(x))}{(x^2+1)^2} \\ &= \frac{(\cos(x^2) \cdot 2x + 2\sin(x)\cos(x))(x^2+1) - 2x(\sin(x^2) + \sin^2(x))}{(x^2+1)^2}.\end{aligned}$$

Problem 7 (Bonus)**.** (a) Compute

$$\frac{d}{dx} \sqrt{\frac{\sqrt{x}+1}{(\cos x + 1)^2}}$$

Solution:

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)' \\ &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2 - 2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}\end{aligned}$$

(b) Find

$$\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1).$$

Solution:

$$\begin{aligned}\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1) &= 4\tan^3(\sqrt[3]{x^5+x^3+2}+1)\cdot\sec(\sqrt[3]{x^5+x^3+2}+1) \\ &\quad \cdot\tan(\sqrt[3]{x^5+x^3+2}+1)\cdot(\sqrt[3]{x^5+x^3+2}+1)' \\ &= 4\tan^4(\sqrt[3]{x^5+x^3+2}+1)\sec(\sqrt[3]{x^5+x^3+2}+1) \\ &\quad \cdot\left(\frac{1}{3}(x^5+x^3+1)^{-2/3}\cdot(5x^4+3x^2)\right).\end{aligned}$$

Problem 8 (Bonus)**.** Calculate

$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$