Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 5
Due Monday, September 30

This week's mastery quiz has four topics. Everyone should submit on M2 and S3. (Even if you got a 2/2 on M2 last week, your total score will still be 2/4 and you should do it again.) If you have a 4/4 on M1 in Blackboard, you don't need to submit it again, and if you have a 2/2 on S2 you don't need to submit that again.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 2: Definition of Derivative
- Secondary Topic 3: Linear Approximation

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim_{x\to 1} \frac{\sin(3x-3)\sin(x-1)}{(x-1)^2} =$

Solution:

$$\lim_{x\to 1} \frac{\sin(3x-3)\sin(x-1)}{(x-1)^2} = \lim_{x\to 1} \frac{\sin(3x-3)}{x-1}\frac{\sin(x-1)}{x-1}$$
$$= \lim_{x\to 1} 3\frac{\sin(3x-3)}{3x-3} = 3.$$

(b) $\lim_{x\to 5} \frac{x^2+x+1}{x^2-10x+25} =$

Solution: The limit of the top is 31 and the limit of the bottom is 0. Furthermore the bottom is $(x-5)^2 \geq 0$, and thus the fraction is always positive; so the limit is in fact $+\infty$.

(c) $\lim_{x\to -2} \frac{x+2}{\sqrt{x+6}-2} =$

Solution:

$$\lim_{x\to -2} \frac{x+2}{\sqrt{x+6}-2} = \lim_{x\to -2} \frac{(x+2)(\sqrt{x+6}+2)}{x+6-4}$$
$$= \lim_{x\to -2} \frac{\sqrt{x+6}+2}{1} = 4.$$

Major Topic 2: Computing Derivatives

(a) $\frac{d}{dx}\sec(\tan(\cos((x+1)^2)))$

Solution:

$\sec(\tan(\cos((x+1)^2)))\tan(\tan(\cos((x+1)^2)))\sec^2(\cos((x+1)^2))(-\sin((x+1)^2))2(x+1)$

(b) Compute $\frac{d}{dx}\frac{\sin(\csc(x^2+1))}{x^4+\cos(x)} =$

Solution:

$$\frac{\big(\cos(\csc(x^2+1))(-\csc(x^2+1)\cot(x^2+1))2x\big)(x^4+\cos(x)) - (4x^3-\sin(x))\sin(\csc(x^2+1))}{(x^4+\cos(x))^2}.$$

Secondary Topic 2: Definition of Derivative

(a) If $f(x) = \frac{x+1}{x-1}$, find $f'(2)$ directly from the formal definition of the derivative.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3+h}{1+h} - 3}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\frac{3+h-3(1+h)}{1+h} \\ &= \lim_{h\to 0} \frac{-2h}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-2}{1+h} = -2. \end{aligned}$$

(b) If $g(x) = x^3 - 3x$, find $g'(x)$, directly from the definition of derivative.

Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - 3(x+h) - x^3 + 3x}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h - x^3 + 3x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 3 = 3x^2 - 3. \end{aligned}$$

Secondary Topic 3: Linear Approximation

(a) Use linear approximation to estimate $\sqrt[4]{14}$.

Solution: We take $f(x) = \sqrt[4]{x}$, and take $a = 16$. Then

$$\begin{aligned} f'(x) &= \frac{1}{4}x^{-3/4} \\ f'(16) &= \frac{1}{4}(16)^{-3/4} = \frac{1}{4\cdot 8} = \frac{1}{32} \\ f(x) &\approx f(a) + f'(a)(x-a) = 2 + \frac{1}{32}(14-16) = 2 - \frac{1}{16} = \frac{31}{16}. \end{aligned}$$

(b) Give a formula for a linear approximation to $g(x) = \sin(x^2 - 3x)$ near the point $a = 0$.

Solution:

$$\begin{aligned}
g'(x) &= \cos(x^2 - x)(2x - 3) \\
g'(0) &= 1 \cdot (0 - 3) = -3 \\
g(x) &\approx 0 - 3(x - 0) \\
&= -3x.
\end{aligned}$$