Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 5
Due Wednesday, October 2

This week's mastery quiz has four topics. Everyone should submit M2, S3, and S4. If you have a 4/4 on M1 (meaning you've gotten two different 2s). If you're unsure about your current grade, please check Blackboard.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 3: Linear Approximation
- Secondary Topic 4: Rates of Change

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim_{x \to +\infty} \frac{3x^2+2x+1}{\sqrt{x^4-x^2+x}} =$

Solution:

$$\begin{aligned}\lim_{x \to +\infty} \frac{3x^2+2x+1}{\sqrt{x^4-x^2+x}} &= \lim_{x \to +\infty} \frac{3+2/x+1/x^2}{1/\sqrt{x^4}\sqrt{x^4-x^2+x}} \\ &= \lim_{x \to +\infty} \frac{3+2/x+1/x^2}{\sqrt{1-1/x^2+1/x^3}} \\ &= \frac{3+0+0}{\sqrt{1-0+0}} = 3.\end{aligned}$$

(b) $\lim_{x \to 4^+} \frac{x+1}{x-4} =$

Solution: The limit of the top is 5 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom will always be positive as we approach from the right, the overall limit is in fact $+\infty$.

(c) $\lim_{x \to 2} \frac{1}{x-2} - \frac{1}{x^2-3x+2} =$

Solution:

$$\begin{aligned}\lim_{x \to 2} \frac{1}{x-2} - \frac{1}{x^2-3x+2} &= \lim_{x \to 2} \frac{x-1}{(x-1)(x-2)} - \frac{1}{(x-1)(x-2)} \\ &= \lim_{x \to 2} \frac{x-2}{(x-1)(x-2)} = \lim_{x \to 2} \frac{1}{x-1} = 1.\end{aligned}$$

Major Topic 2: Computing Derivatives

(a) Compute $\frac{d}{dx} \cos^2\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)$.

Solution:

$$\begin{aligned}&2\cos\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\left(-\sin\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\right) \\ &\cdot 2\tan\left(\sec^2(\sqrt{x}+x)\right)\sec^2\left(\sec^2(\sqrt{x}+x)\right) \\ &\cdot 2\sec(\sqrt{x}+x)\sec(\sqrt{x}+x)\tan(\sqrt{x}+x)\left(\frac{1}{2x}+1\right)\end{aligned}$$

(b) $\frac{d}{dx} \sec\left(\frac{x^2+1}{\sqrt{x^3-2}}\right) =$

Solution:

$$\sec\left(\frac{x^2+1}{\sqrt{x^3-2}}\right)\tan\left(\frac{x^2+1}{\sqrt{x^3-2}}\right)\frac{2x\sqrt{x^3-2}-(x^2+1)\frac{1}{2}(x^3-2)^{-1/2}\cdot 3x^2}{x^3-2}.$$

Secondary Topic 3: Linear Approximation

(a) Find a linear approximation to the function $f(x)=\frac{x^3}{1+x}$ near the point $a=1$ and use it to approximate $f(1.3)$.

Solution:

$$f(1)=\frac{1}{2}$$
$$f'(x)=\frac{3x^2(1+x)-x^3}{(1+x)^2}$$
$$f'(1)=\frac{6-1}{4}=\frac{5}{4}$$
$$f(x)\approx\frac{1}{2}+\frac{5}{4}(x-1)$$
$$f(1.3)\approx\frac{1}{2}+\frac{5}{4}\cdot .3=\frac{20}{40}+\frac{15}{40}=\frac{35}{40}=\frac{7}{8}.$$

(b) Find a formula for a linear approximation to $g(x)=\frac{x+1}{x-1}$ near the point $a=2$.

Solution:

$$g'(x)=\frac{(x-1)-(x+1)}{(x-1)^2}$$

and thus at $x=2$ we have $g'(2)=\frac{1-3}{1^2}=-2$. We compute $g(2)=3$ so we have the equation

$$g(x)\approx 3-2(x-2).$$

Secondary Topic 4: Rates of change

(a) Suppose the vertical position of a weight on a spring in inches is given as a function of time in seconds by the formula $h(t)=\cos(2t)$.

(i) When is the velocity zero?

(ii) When is the acceleration zero?

Solution:

(i) $p'(t) = -2\sin(2t)$ so the velocity is zero when $\sin(2t) = 0$. This happens when $2t = 0, \pi, 2\pi, \ldots$, and thus when $t = 0, \pi/2, \pi, 3\pi/2, \ldots$. In other words, at $n\pi/2$.

(ii) $p''(t) = -4\cos(2t)$ is zero when $2t = \pi/2, 3\pi/2, \ldots$, and thus when $t = \pi/4, 3\pi/4, 5\pi/4, \ldots$. We could say $t = (2n+1)\pi/4$.

(b) The *area moment of inertia* of a steel beam measures how difficult it is to bend, and is measured in m^4. If a square beam has a side length of s meters, then its moment of inertia is given by $L(s) = s^4/12$.

(i) What are the units of $L'(s)$? What does it represent physically? What does it mean if L' is big?

(ii) Compute $L'(2)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(i) $L'(s)$ has units of $\frac{m^4}{m} = \mathrm{m}^3$. It describes how much increasing the side length by a meter would increase the area moment of inertia. If it's large, that means that making the beam a little longer will increase the moment of inertia by a lot.

(ii) $L'(s) = 4s^3/12 = s^3/3$ so $L'(2) = 8/3 = 8/3\mathrm{m}^3$. This tells us that if we increase the side length by one meter from 2 meters to 3 meters, we should increase the moment of inertia by about $8/3\mathrm{m}^4$.