Math 1231-13: Single-Variable Calculus 1
George Washington University   Fall 2024
Recitation 5

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Friday September 27, 2024

Problem 1 (Bonus)**.** Consider the function $\sec^2\left(x^2+1\right)$

(a) Find functions f and g so that $(f \circ g)(x) = \sec^2\left(x^2+1\right)$.

(b) Talk to the people next to you. Did they pick the same f and g that you did? Can you find a different pair of functions f and g that also work?

(c) Find functions f, g, h so that $(f \circ g \circ h)(x) = \sec^2\left(x^2+1\right)$.

(d) Compute f', g', and h'.

(e) What is $\frac{d}{dx}\sec^2\left(x^2+1\right)$?

Solution:

(a) There are basically two choices here. You could say that $f(x) = \sec^2(x)$ and $g(x) = x^2+1$, which is maybe the more obvious choice; or you could say that $f(x) = x^2$ and $g(x) = \sec(x^2+1)$.

(b) This is really a composite of three functions, which is why you could make different choices here.

(c) $f(x) = x^2, g(x) = \sec(x), h(x) = x^2+1$. (Technically there are other things you could do, like $g(x) = \sec(x+1)$ and $h(x) = x^2$, but those are moderately silly.)

(d) $f'(x) = 2x, g'(x) = \sec(x)\tan(x), h'(x) = 2x$.

(e)

$$\begin{aligned}
\frac{d}{dx}\sec^2\left(x^2+1\right) &= f'(g(h(x))\cdot g'(h(x))\cdot h'(x) \\
&= f'(\sec(x^2+1))\cdot g'(x^2+1)\cdot h'(x) \\
&= 2\sec(x^2+1)\cdot\sec(x^2+1)\tan(x^2+1)\cdot 2x.
\end{aligned}$$

Problem 2. Find

$$\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1}$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1} &= \frac{(\sin(x^2)+\sin^2(x))'(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2} \\
&= \frac{(\cos(x^2)\cdot 2x+2\sin(x)\cos(x))(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2}.
\end{aligned}$$

Problem 3. (a) Compute

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}}$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)' \\
&= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2-2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}
\end{aligned}$$

(b) Find

$$\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1).$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1) &= 4\tan^3(\sqrt[3]{x^5+x^3+2}+1)\cdot\sec(\sqrt[3]{x^5+x^3+2}+1) \\
&\quad\cdot\tan(\sqrt[3]{x^5+x^3+2}+1)\cdot(\sqrt[3]{x^5+x^3+2}+1)' \\
&= 4\tan^4(\sqrt[3]{x^5+x^3+2}+1)\sec(\sqrt[3]{x^5+x^3+2}+1) \\
&\quad\cdot\left(\frac{1}{3}(x^5+x^3+1)^{-2/3}\cdot(5x^4+3x^2)\right).
\end{aligned}$$

Problem 4 (Bonus)**.** Calculate

$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$

Problem 5 (Geometric Series)**.** One function it's sometimes important to approximate is the "geometric series" formula $f(x) = \frac{1}{1-x}$, near $x = 0$.

(a) What is $f'(x)$?

(b) Find a linear approximation for $f(x)$ near $x = 0$.

(c) Use this formula to estimate $\frac{1}{.9}$ and $\frac{1}{1.01}$. Do these answers make sense?

(d) Use your formula to estimate $\frac{1}{1.5}$ and $frac10.5$. Do these answers make sense?

(e) Use your formula to estimate $f(-1)$ and $f(1)$. Do these answers make sense?

Solution:

(a) $f'(x) = -(1-x)^{-2} = \frac{1}{(1-x)^2}$. This is tricky; you get a negative sign from the power rule, but another from the chain rule that cancels it out.

(This is a weird way to write the function! Why not just use $\frac{1}{1+x}$? Because this setup makes more sense in a lot of the applications people want to use it for. You'll see why when you study power series in Calculus 2.)

(b) $f'(0) = 1$, so our linear approximation is $f(x) \approx 1 + x$.

(c) $\frac{1}{.9} = f(.1) \approx 1.1$. The true answer is $1.\overline{11}$, so that checks out.

$\frac{1}{1.01} = f(-.01) \approx .99$. The true answer is $.990099\ldots$, which also makes sense.

(d) $\frac{1}{1.5} = f(-0.5) \approx 0.5$. The true answer is $2/3 \approx .\overline{66}$ so this is, like, okay-ish.

$\frac{1}{0.5} = f(0.5) \approx 1.5$. The true answer is 2, so this is again okay, but not great.

(e) $f(-1) \approx 0$. But $f(-1) = 1/2$, so that doesn't make a ton of sense. This is because (-1) is "far away" from zero for our purposes. And how do we know it's far away? Well...

$f(1) \approx 0$. But $f(1)$ is utterly undefined, since it asks us to divide by 0. We've gone too far away for the linear approximation to work at all.

Problem 6. (a) Use the binomial approximation to estimate $\sqrt{2}$ and $\sqrt[n]{2}$.

(b) Use the binomial approximation to estimate $\sqrt{17}$. (Remember: 17 is not close to 1! You need to be slightly clever here.)

(c) Can you find a formula to approximate $(1+x^n)^\alpha$ for a real number α?

(d) What does this tell us about $\sqrt{1+x^2}$?

Solution:

(a) We have $\sqrt{2} = (1+1)^{1/2} \approx 1 + \frac{1}{2} \cdot 1 \approx 3/2$, and $\sqrt[n]{2} = (1+1)^{1/n} \approx 1 + \frac{1}{n} = \frac{n+1}{n}$.

(b)

$$\begin{aligned}\sqrt{17} &= \sqrt{16 \cdot 17/16} = 4\sqrt{17/16} = 4\sqrt{1+1/16} \\ &= 4(1+1/16)^{1/2} \approx 4\left(1 + \frac{1}{32}\right) \\ &= 4 + \frac{1}{8} = 4.125.\end{aligned}$$

The true answer is $4.12311\ldots$.

(c) By the binomial approximation, $(1+x^n)^\alpha \approx 1 + \alpha x^n$.

(d) Thus in particular, $\sqrt{1+x^2} \approx 1 + \frac{1}{2}x^2$. Note that this is very different from $1+x$!

Problem 7 (Bonus)**.** Find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

Solution: We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.