Math 1231: Single-Variable Calculus 1
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Recitation 5

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Problem 1 (Geometric Series)**.** One function it's sometimes important to approximate is the "geometric series" formula $f(x) = \dfrac{1}{1-x}$, near $x = 0$.

(a) What is $f'(x)$?

(b) Find a linear approximation for $f(x)$ near $x = 0$.

(c) Use this formula to estimate $\frac{1}{.9}$ and $\frac{1}{1.01}$. Do these answers make sense?

(d) Use your formula to estimate $\frac{1}{1.5}$ and $frac10.5$. Do these answers make sense?

(e) Use your formula to estimate $f(-1)$ and $f(1)$. Do these answers make sense?

Solution:

(a) $f'(x) = -(1-x)^{-2} = \frac{1}{(1-x)^2}$. This is tricky; you get a negative sign from the power rule, but another from the chain rule that cancels it out.

(This is a weird way to write the function! Why not just use $\frac{1}{1+x}$? Because this setup makes more sense in a lot of the applications people want to use it for. You'll see why when you study power series in Calculus 2.)

(b) $f'(0) = 1$, so our linear approximation is $f(x) \approx 1 + x$.

(c) $\frac{1}{.9} = f(.1) \approx 1.1$. The true answer is $1.\overline{11}$, so that checks out.

$\frac{1}{1.01} = f(-.01) \approx .99$. The true answer is $.990099\ldots$, which also makes sense.

(d) $\frac{1}{1.5} = f(-0.5) \approx 0.5$. The true answer is $2/3 \approx .\overline{66}$ so this is, like, okay-ish.

$\frac{1}{0.5} = f(0.5) \approx 1.5$. The true answer is 2, so this is again okay, but not great.

(e) $f(-1) \approx 0$. But $f(-1) = 1/2$, so that doesn't make a ton of sense. This is because (-1) is "far away" from zero for our purposes. And how do we know it's far away? Well...

$f(1) \approx 0$. But $f(1)$ is utterly undefined, since it asks us to divide by 0. We've gone too far away for the linear approximation to work at all.

Problem 2. (a) Use the binomial approximation to estimate $\sqrt{2}$ and $\sqrt[n]{2}$.

(b) Use the binomial approximation to estimate $\sqrt{17}$. (Remember: 17 is not close to 1! You need to be slightly clever here.)

(c) Can you find a formula to approximate $(1+x^n)^\alpha$ for a real number α?

(d) What does this tell us about $\sqrt{1+x^2}$?

Solution:

(a) We have $\sqrt{2} = (1+1)^{1/2} \approx 1 + \frac{1}{2} \cdot 1 \approx 3/2$, and $\sqrt[n]{2} = (1+1)^{1/n} \approx 1 + \frac{1}{n} = \frac{n+1}{n}$.

(b)

$$\begin{aligned}\sqrt{17} &= \sqrt{16 \cdot 17/16} = 4\sqrt{17/16} = 4\sqrt{1+1/16} \\ &= 4(1+1/16)^{1/2} \approx 4\left(1 + \frac{1}{32}\right) \\ &= 4 + \frac{1}{8} = 4.125.\end{aligned}$$

The true answer is $4.12311\ldots$.

(c) By the binomial approximation, $(1+x^n)^\alpha \approx 1 + \alpha x^n$.

(d) Thus in particular, $\sqrt{1+x^2} \approx 1 + \frac{1}{2}x^2$. Note that this is very different from $1 + x$!

Problem 3 (Bonus)**.** Find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

Solution: We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.

Problem 4. Suppose a particle has height as a function of time given by $h(t\mathrm{s}) = (2t^3 - 3t^2 - 12t + 3)\,\mathrm{m}$.

(a) What is the velocity of this particle at time $t = 0$? What are the units, and why?

(b) What is the acceleration of this particle at time $t = 0$? What are the units and why?

(c) When is the particle speeding up? When is it slowing down?

Solution:

(a) $h'(t) = (6t^2 - 6t - 12)\,\mathrm{m/s}$ so $h'(0) = -12\mathrm{m/s}$. We get m/s because the input is in seconds and the output is in meters, so the derivative, which is $\frac{\Delta h}{\delta t}$, is meters over seconds.

(b) $h''(t) = (12t - 6)\,\mathrm{m/s^2}$ so $h''(0) = 12\mathrm{m/s^2}$. Here, the derivative is $\frac{\Delta h'}{\Delta t}$, so the numerator is in m/s and the denominator is in s, giving m/s^2.

(c) The particle is speeding up when the derivative is increasing. This would have to mean the second derivative is positive, and thus we want $12t > 6$ and so $t > 1/2$. The particle is slowing down when $t < 1/2$.

Problem 5. Suppose that $p(t) = 10 - 2t$ is momentum (in kg m/s) of a ball thrown directly upwards, as a function of time (in seconds).

(a) What units does the derivative $p'(t)$ take as input? What units are its output? (Do you know of any physical quantity that's represented by those units?)

(b) What does the derivative $p'(t)$ represent physically? What would it mean for $p'(t)$ to be big, or small?

(c) Calculate $p'(3)$. What does this tell you? What physical observation could you measure to check if your calculation was correct?

Solution:

(a) The original function p takes in time (in seconds) and outputs momentum (in kg m/s). So the derivative takes in time in seconds, and outputs momentum per second in kg m/s^2.

(You may notice that this is also the units of mass times acceleration, which is just force! The original formulation of Newton's second law was $F = \frac{dp}{dt}$. Which means that, just like the derivative of velocity is acceleration, the derivative of momentum is force.)

(b) The derivative is the rate at which the momentum changes/decreases over time. If $p'(t)$ is large and positive the momentum is increasing quickly; if it's large and negative the momentum is decreasing quickly; and if it's close to zero, the momentum isn't changing much.

(c) $p'(t) = -2$ so $p'(3) = -2$. This means that the momentum decreases by about 2 kilogram meters per second every second.

If we want to check this, we could measure the momentum of our ball at time $t = 3$ and again at time $t = 4$. Between those two times, the momentum will decrease by about 2 kilogram-meters per second.

(The derivative is constant, but that does *not* mean that the momentum doesn't change. It means the rate at which the momentum is changing doesn't change, but that is importantly different.)

Problem 6. Suppose the cost of buying m machines is $C(m) = 500 + 10m + .05m^2$. There's some start-up cost to having any machines at all; then each machine costs a bit more than the previous one.

(a) What are the units of the inputs to the function C? What are the units of the outputs?

(b) What is $C(1)$? $C(10)$? $C(100)$?

(c) Find a formula for $C'(m)$. What are the units of the input and output to $C'(m)$?

(d) What is $C'(10)$? How should we interpret this number?

(e) What is the *average* cost per machine when you have ten machines? How does this compare to your previous answer?

(f) What is $C''(m)$? What are the units? What is $C''(10)$ and how should we interpret it?

Solution:

(a) The units of the input are "machines" and the units of output are "dollars".

(b) We can see that $C(1) = \$510.05$, and $C(10) = \$605$. $C(100) = \$2000$.

(c) $C'(m) = 10 + .1m$. The input to this is still machines, and the output is in dollars per machine.

(d) $C'(m) = 10 + .1m$. The input is machines, and the output is dollars per machine.

(e) $C'(10) = 11$ dollars per machine. This means that if we have ten machines and buy one more, we will have to spend 11 more dollars. (Indeed, $C(11) = 616.05$ which is about 11 more than $C(10) = 605$.

(f) We know that $C(10) = 605$, so the average cost per machine is \$60.5. That's very different from the marginal cost; adding one more machine will only cost \$11 more.

(g) $C''(m) = .1$. This takes in machines and outputs dollars per machine per machine. We see that $C''(10) = .1$ dollars per machine per machine. It tells us that each new machine is going to cost ten cents more than the previous machine did.

Problem 7 (Bonus)**.** Let $Q(p) = 10000 - 10p$ give the number of widgets you can sell at a given price p.

(a) If you set a price of \$100, how many widgets will you be able to sell? What if you set a price of \$1000?

(b) What is the derivative of Q? What are its units?

(c) What is $Q'(100)$ and what does that tell you?

Solution:

(a) $Q(100) = 10000 - 1000 = 9000$ widgets, and $Q(1000) = 0$ widgets. So if you set a price of \$100 you'll be able to sell 9000 widgets, but if you set a price of \$1000 you won't be able to sell any at all.

(b) $Q'(p) = -10$. This takes in dollars, and outputs widgets per dollar.

(c) $Q'(100) = -10$ widgets per dollar. This means that if we raise the price by one dollar, we will sell ten fewer widgets.

(Economists call this the Price Elasticity of Demand: “elasticity” is how quickly one thing responds to changes in another thing. So any time the term “elasticity” shows up in economics, there’s a derivative inolved somewhere).