Math 1231-13: Single-Variable Calculus 1
George Washington University   Spring 2024
Recitation 6

Jay Daigle

Friday February 23, 2024

Problem 1. Suppose a particle has height as a function of time given by $h(t\mathrm{s}) = (2t^3 - 3t^2 - 12t + 3)\,\mathrm{m}$.

(a) What is the velocity of this particle at time $t = 0$? What are the units, and why?

(b) What is the acceleration of this particle at time $t = 0$? What are the units and why?

(c) When is the particle speeding up? When is it slowing down?

Solution:

(a) $h'(t) = (6t^2 - 6t - 12)\,\mathrm{m/s}$ so $h'(0) = -12\mathrm{m/s}$. We get m/s because the input is in seconds and the output is in meters, so the derivative, which is $\frac{\Delta h}{\delta t}$, is meters over seconds.

(b) $h''(t) = (12t - 6)\,\mathrm{m/s^2}$ so $h''(0) = 12\mathrm{m/s^2}$. Here, the derivative is $\frac{\Delta h'}{\Delta t}$, so the numerator is in m/s and the denominator is in s, giving $\mathrm{m/s^2}$.

(c) The particle is speeding up when the derivative is increasing. This would have to mean the second derivative is positive, and thus we want $12t > 6$ and so $t > 1/2$. The particle is slowing down when $t < 1/2$.

Problem 2. Suppose that $p(t) = 10 - 2t$ is momentum (in kg m/s) of a ball thrown directly upwards, as a function of time (in seconds).

(a) What units does the derivative $p'(t)$ take as input? What units are its output? (Do you know of any physical quantity that's represented by those units?)

(b) What does the derivative $p'(t)$ represent physically? What would it mean for $p'(t)$ to be big, or small?

(c) Calculate $p'(3)$. What does this tell you? What physical observation could you measure to check if your calculation was correct?

Solution:

(a) The original function p takes in time (in seconds) and outputs momentum (in $\mathrm{kg\,m/s}$). So the derivative takes in time in seconds, and outputs momentum per second in $\mathrm{kg\,m/s^2}$.

(You may notice that this is also the units of mass times acceleration, which is just force! The original formulation of Newton's second law was $F = \frac{dp}{dt}$. Which means that, just like the derivative of velocity is acceleration, the derivative of momentum is force.)

(b) The derivative is the rate at which the momentum changes/decreases over time. If $p'(t)$ is large and positive the momentum is increasing quickly; if it's large and negative the momentum is decreasing quickly; and if it's close to zero, the momentum isn't changing much.

(c) $p'(t) = -2$ so $p'(3) = -2$. This means that the momentum decreases by about 2 kilogram meters per second every second.

If we want to check this, we could measure the momentum of our ball at time $t = 3$ and again at time $t = 4$. Between those two times, the momentum will decrease by about 2 kilogram-meters per second.

(The derivative is constant, but that does *not* mean that the momentum doesn't change. It means the rate at which the momentum is changing doesn't change, but that is importantly different.)

Problem 3. Suppose the cost of buying m machines is $C(m) = 500 + 10m + .05m^2$. There's some start-up cost to having any machines at all; then each machine costs a bit more than the previous one.

(a) What are the units of the inputs to the function C? What are the units of the outputs?

(b) What is $C(1)$? $C(10)$? $C(100)$?

(c) Find a formula for $C'(m)$. What are the units of the input and output to $C'(m)$?

(d) What is $C'(10)$? How should we interpret this number?

(e) What is the *average* cost per machine when you have ten machines? How does this compare to your previous answer?

(f) What is $C''(m)$? What are the units? What is $C''(10)$ and how should we interpret it?

Solution:

(a) The units of the input are "machines" and the units of output are "dollars".

(b) We can see that $C(1) = \$510.05$, and $C(10) = \$605$. $C(100) = \$2000$.

(c) $C'(m) = 10 + .1m$. The input to this is still machines, and the output is in dollars per machine.

(d) $C'(m) = 10 + .1m$. The input is machines, and the output is dollars per machine.

(e) $C'(10) = 11$ dollars per machine. This means that if we have ten machines and buy one more, we will have to spend 11 more dollars. (Indeed, $C(11) = 616.05$ which is about 11 more than $C(10) = 605$.

(f) We know that $C(10) = 605$, so the average cost per machine is \$60.5. That's very different from the marginal cost; adding one more machine will only cost \$11 more.

(g) $C''(m) = .1$. This takes in machines and outputs dollars per machine per machine. We see that $C''(10) = .1$ dollars per machine per machine. It tells us that each new machine is going to cost ten cents more than the previous machine did.

Problem 4 (Bonus)**.** Let $Q(p) = 10000 - 10p$ give the number of widgets you can sell at a given price p.

(a) If you set a price of \$100, how many widgets will you be able to sell? What if you set a price of \$1000?

(b) What is the derivative of Q? What are its units?

(c) What is $Q'(100)$ and what does that tell you?

Solution:

(a) $Q(100) = 10000 - 1000 = 9000$ widgets, and $Q(1000) = 0$ widgets. So if you set a price of \$100 you'll be able to sell 9000 widgets, but if you set a price of \$1000 you won't be able to sell any at all.

(b) $Q'(p) = -10$. This takes in dollars, and outputs widgets per dollar.

(c) $Q'(100) = -10$ widgets per dollar. This means that if we raise the price by one dollar, we will sell ten fewer widgets.

(Economists call this the Price Elasticity of Demand: "elasticity" is how quickly one thing responds to changes in another thing. So any time the term "elasticity" shows up in economics, there's a derivative inolved somewhere).

Problem 5. Find an equation for the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

Solution: We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.

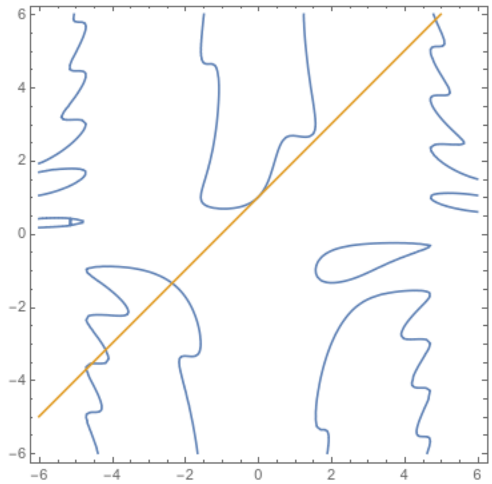
Problem 6. Find an equation for the tangent line to $y\cos(x) = 1 + \sin(xy)$ at the point $(0, 1)$.

Solution: We can compute

$$\frac{d}{dx}\left(y\cos(x)\right) = \frac{d}{dx}\left(1 + \sin(xy)\right)$$
$$\frac{dy}{dx}\cos(x) - y\sin(x) = \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

At this point we could simplify this expression, but we *shouldn't*. Instead, we can plug in $(0, 1)$ now, and get

$$\frac{dy}{dx}\cos(0) - 1\sin(0) = \cos(0)\left(1 + 0\frac{dy}{dx}\right)$$
$$\frac{dy}{dx} = 1$$
$$y - 1 = 1(x - 0)$$
$$y = x + 1.$$



Now, we *could* find a formula for y' in terms of x and y; that would give something like this.

$$\frac{dy}{dx}\left(\cos(x) - x\cos(xy)\right) = y\cos(xy) + y\sin(x)$$
$$\frac{dy}{dx} = \frac{y\cos(xy) + y\sin(x)}{\cos(x) - x\cos(xy)}.$$

But we don't need to for the question that we asked.

Problem 7. Suppose we have some function f such that $8f(x) + x^2(f(x))^3 = 24$, and we know that $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Now suppose we want to estimate $f(5)$ (without having to solve the equation).

(a) Use implicit differentiation on the equation $8f(x) + x^2(f(x))^3 = 24$ to find a formula relating $x, f(x)$ and $f'(x)$.

(b) Use this formula to determine $f'(4)$.

(c) Find a formula for the linear approximation of f near 4.

(d) Estimate $f(5)$.

Solution:

(a)

$$\frac{d}{dx}\left(8f(x) + x^2(f(x))^3\right) = \frac{d}{dx}24$$
$$8f'(x) + 2x(f(x))^3 + 3x^2(f(x))^2 f'(x) = 0$$

(b)

$$\begin{aligned}
8f'(4) + 2 \cdot 4 \cdot 1^3 + 3 \cdot 4^2 \cdot 1^2 f'(4) &= 0 \\
8f'(4) + 8 + 48 f'(4) &= 0 \\
f'(4) &= -1/7
\end{aligned}$$

(c) We have a derivative, so we can again compute a linear approximation. We get

$$f(x) \approx f'(4)(x-4) + f(4) = \frac{-1}{7}(x-4) + 1.$$

(d) We compute

$$f(5) \approx \frac{-1}{7}(5-4) + 1 = 1 + \frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close.

Problem 8 (Bonus). (a) If $\sqrt{xy} = x^2 y - 2$, find a formula for $\frac{dy}{dx}$ in terms of x and y.

(b) Find an equation of the tangent line at the point $(1, 4)$.

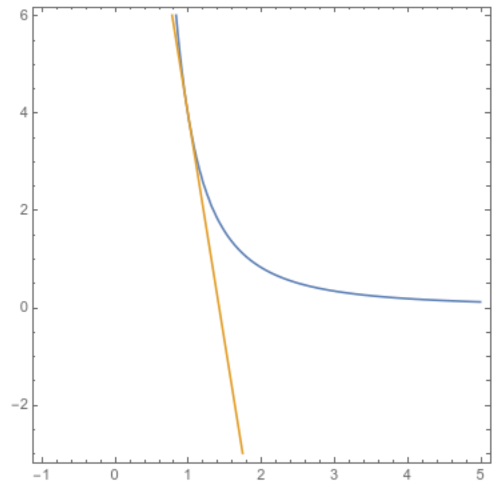
Solution:

(a)

$$\begin{aligned}
\frac{d}{dx}\sqrt{xy} &= \frac{d}{dx}\left(x^2 y - 2\right) \\
\frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) &= 2xy + x^2\frac{dy}{dx} \\
\frac{dy}{dx}\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right) &= \frac{1}{2}(xy)^{-1/2}y - 2xy \\
\frac{dy}{dx} &= \frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{x^2 - \frac{1}{2}x(xy)^{-1/2}}.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{dy}{dx}(1,4) &= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot 4 - 8}{1 - \frac{1}{2}\cdot\frac{1}{2}} = \frac{-7}{3/4} = \frac{-28}{3} \\
y - 4 &= \frac{-28}{3}(x-1).
\end{aligned}$$


6
4
2
0
−2
−1
0
1
2
3
4
5