# Math 1231: Single-Variable Calculus 1
## George Washington University  Fall 2024
## Recitation 6

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**Problem 1.** Find an equation for the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

**Solution:** We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.

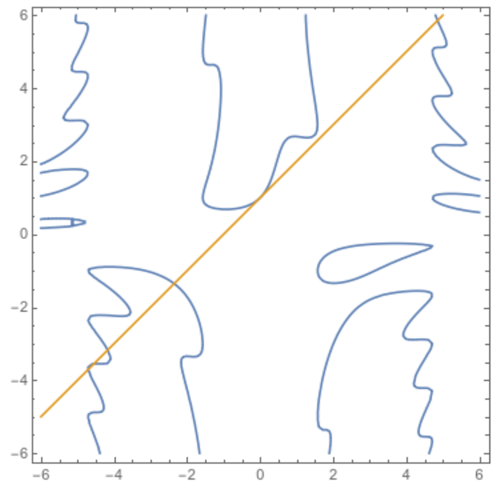
**Problem 2.** Find an equation for the tangent line to $y\cos(x) = 1 + \sin(xy)$ at the point $(0, 1)$.

**Solution:** We can compute

$$\begin{aligned}
\frac{d}{dx}\left(y\cos(x)\right) &= \frac{d}{dx}\left(1 + \sin(xy)\right) \\
\frac{dy}{dx}\cos(x) - y\sin(x) &= \cos(xy)\left(y + x\frac{dy}{dx}\right)
\end{aligned}$$

At this point we could simplify this expression, but we *shouldn't*. Instead, we can plug in $(0, 1)$ now, and get

$$\begin{aligned}
\frac{dy}{dx}\cos(0) - 1\sin(0) &= \cos(0)\left(1 + 0\frac{dy}{dx}\right) \\
\frac{dy}{dx} &= 1 \\
y - 1 &= 1(x - 0) \\
y &= x + 1.
\end{aligned}$$




Now, we *could* find a formula for $y'$ in terms of $x$ and $y$; that would give something like this.

$$\frac{dy}{dx}\left(\cos(x) - x\cos(xy)\right) = y\cos(xy) + y\sin(x)$$
$$\frac{dy}{dx} = \frac{y\cos(xy) + y\sin(x)}{\cos(x) - x\cos(xy)}.$$

But we don't need to for the question that we asked.

**Problem 3.** Suppose we have some function $f$ such that $8f(x) + x^2(f(x))^3 = 24$, and we know that $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Now suppose we want to estimate $f(5)$ (without having to solve the equation).

(a) Use implicit differentiation on the equation $8f(x) + x^2(f(x))^3 = 24$ to find a formula relating $x, f(x)$ and $f'(x)$.

(b) Use this formula to determine $f'(4)$.

(c) Find a formula for the linear approximation of $f$ near 4.

(d) Estimate $f(5)$.

**Solution:**

(a)

$$\frac{d}{dx}\left(8f(x) + x^2(f(x))^3\right) = \frac{d}{dx}24$$
$$8f'(x) + 2x(f(x))^3 + 3x^2(f(x))^2 f'(x) = 0$$

(b)

$$\begin{aligned} 8f'(4) + 2 \cdot 4 \cdot 1^3 + 3 \cdot 4^2 \cdot 1^2 f'(4) &= 0 \\ 8f'(4) + 8 + 48f'(4) &= 0 \\ f'(4) &= -1/7 \end{aligned}$$

(c) We have a derivative, so we can again compute a linear approximation. We get

$$f(x) \approx f'(4)(x-4) + f(4) = \frac{-1}{7}(x-4) + 1.$$

(d) We compute

$$f(5) \approx \frac{-1}{7}(5-4) + 1 = 1 + \frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close.

**Problem 4** (Bonus). (a) If $\sqrt{xy} = x^2y - 2$, find a formula for $\frac{dy}{dx}$ in terms of $x$ and $y$.

(b) Find an equation of the tangent line at the point $(1, 4)$.

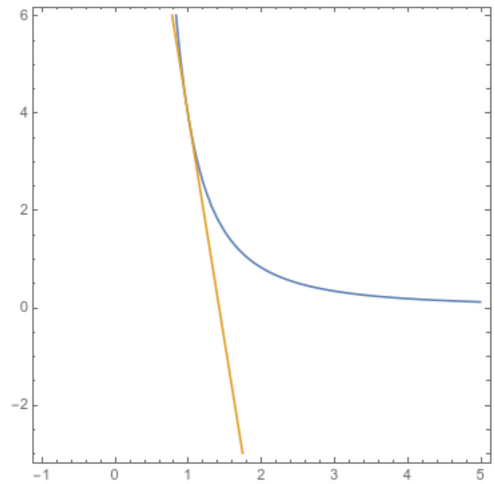
**Solution:**

(a)

$$\begin{aligned} \frac{d}{dx}\sqrt{xy} &= \frac{d}{dx}\left(x^2y - 2\right) \\ \frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) &= 2xy + x^2\frac{dy}{dx} \\ \frac{dy}{dx}\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right) &= \frac{1}{2}(xy)^{-1/2}y - 2xy \\ \frac{dy}{dx} &= \frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{x^2 - \frac{1}{2}x(xy)^{-1/2}}. \end{aligned}$$

(b)

$$\begin{aligned} \frac{dy}{dx}(1,4) &= \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot 4 - 8}{1 - \frac{1}{2} \cdot \frac{1}{2}} = \frac{-7}{3/4} = \frac{-28}{3} \\ y - 4 &= \frac{-28}{3}(x-1). \end{aligned}$$




**Problem 5** (Bonus)**.** A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) What equation should we use here, and why?

(d) Use a derivative to calculate the answer to the question. Does your answer make sense?

(e) To check things: how long and wide will the rectangle be after one inch? How much will the area have increased? Does that make sense with your answer to the related rates problem?

(f) Bonus: where have we seen basically this argument before?

**Solution:**

(a) It's a rectangle.

(b) We want to know how quickly the area is increasing, so we're looking for $\frac{dA}{dt}$, and the units should be in$^2$/s.

(c) We can relate all our quantities with the formula for the area of a rectangle: $A = \ell w$ relates the area, which we want to know about, to the length and width, which we do know about.

We have $\ell = 5\text{in}, w = 7\text{in}, \frac{d\ell}{dt} = 1\text{in/s}, \frac{dw}{dt} = 2\text{in/s}$. Taking a derivative gives us

$$\begin{aligned} \frac{dA}{dt} &= \ell\frac{dw}{dt} + w\frac{d\ell}{dt} \\ &= 5\text{in} \cdot 2\text{in/s} + 7\text{in} \cdot 1\text{in/s} \\ &= 17\text{in}^2/\text{s}. \end{aligned}$$

The units are right (the rate at which area is changing per second), and the direction is right (the area should be increasing, and this derivative is positive). It's really hard to see if the size is right using our intuition; people in general have bad intuition for the rate at which area changes in response to lengths.

One second later, we'd have $\ell = 6\text{in}$ and $w = 9\text{in}$ for a total area of $54\text{in}^2$. This is an increase of $19\text{in}^2$ over our starting area of $35\text{in}^2$, and 17 is a pretty good approximation of 19.

The derivative of the area formula is just the product rule; we saw basically this same picture during the proof of the product rule.