Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 7
Due Monday, October 14

This week's mastery quiz has four topics. Everyone should submit on S5 and S6. If you have a 4/4 on M2 in Blackboard, you don't need to submit it again, and if you have a 2/2 on S4 you don't need to submit that again. (Note the midterm can improve your mastery scores, so do try to check Blackboard! We'll get the grades up when we can.)

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Secondary Topic 4: Rates of Change
- Secondary Topic 5: Implicit Differentiation
- Secondary Topic 6: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

(a) Compute $\frac{d}{dx} \cos\left(\sec^2\left(x^3 \tan(x^2)\right)\right)$.

Solution:

$$f'(x) = -\sin\left(\sec^2\left(x^3\tan(x^2)\right)\right) \cdot 2\sec\left(x^3\tan(x^2)\right) \cdot \sec\left(x^3\tan(x^2)\right)\tan\left(x^3\tan(x^2)\right) \cdot \left(3x^2\tan(x^2) + x^3\sec^2(x^2)\cdot 2x\right)$$

(b) Compute $\frac{d}{dx}\left(\frac{x\csc(x)}{\sqrt{x^3-x}}\right)^3$

Solution:

$$g'(x) = 3\left(\frac{x\csc(x)}{\sqrt{x^3-x}}\right)^2 \frac{\left(\csc(x) - x\csc(x)\cot(x)\right)\sqrt{x^3-1} - x\csc(x)\frac{1}{2}(x^3-x)^{-1/2}(3x^2-1)}{x^3-x}.$$

Secondary Topic 4: Rates of Change

(a) Let $F(x) = \frac{1}{x} + 1$ be the amount of pressure exerted on a beam in pounds per square inch at a point x inches to the right of its left end.

(i) What are the units of $F'(x)$? What does $F'(x)$ represent physically? What would it mean if $F'(x)$ is big?

(ii) Compute $F'(5)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(i) The derivative $F'(x)$ has units pounds per square inch per inch, or pounds per cubic inch. $F'(x)$ is the rate at which pressure is increasing as you move to the right along the stick. If $F'(x)$ is big, that means that moving along the stick a little bit will increase the pressure by a lot.

(ii) $F'(x) = -1/x^2$ so $F'(5) = -1/25$. This means that if we are five inches to the right of the endpoint, moving one more inch to the right should decrease the pressure by about 1/25 of a pound per square inch.

(b) Suppose the height of a particle in centimeters is given as a function of time in seconds $p(t) = t^3 - 3t$.

(i) When is the velocity zero?

(ii) When is the acceleration zero?

Solution:

(i) $p'(t) = 3t^2 - 3$ is zero when $t = \pm 1$ second.

(ii) $p''(t) = 6t$ is zero when $t = 0$ seconds.

Secondary Topic 5: Implicit Differentiation

(a) Find a formula for $\frac{d^2y}{dx^2}$ if $x^3 = xy + 1$.

Solution:

$$\begin{aligned}
3x^2 &= y + xy' \\
y' &= \frac{3x^2 - y}{x} && = 3x - \frac{y}{x} \\
y'' &= \frac{(6x - y')x - (3x^2 - y)}{x^2} && = 3 - \frac{y'x - y}{x^2} \\
&= \frac{\left(6x - \frac{3x^2 - y}{x}\right)x - (3x^2 - y)}{x^2} && = 3 - \frac{\frac{y}{x} \cdot x - y}{x^2} = 3.
\end{aligned}$$

(b) Find an equation for the line tangent to the curve $x^2y - xy^3 = xy + 3$ at the point $(3, 1)$.

Solution:

$$\begin{aligned}
2xy + x^2y' - y^3 - 3xy^2y' &= y + xy' \\
6 + 9y' - 1 - 9y' &= 1 + 3y' \\
4 &= 3y' \\
y' &= 4/3
\end{aligned}$$

and thus an equation for the tangent line is

$$y - 1 = \frac{4}{3}(x - 3).$$

Secondary Topic 6: Related Rates

(a) A snowball is melting such that its surface area is decreasing at $1\text{cm}^2/\text{min}$. When the radius is 8cm, how quickly is the radius decreasing?

(The surface area of a sphere of radius r is $4\pi r^2$.)

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.

Solution: We have $S = 4\pi r^2$, so $S' = 8\pi r r'$. When the radius is 8cm we have

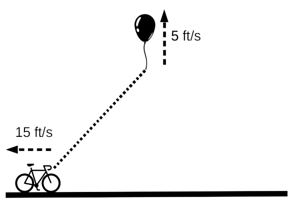
$$\begin{aligned} S' &= 8\pi \cdot 8\text{cm} \cdot r' \\ -1\text{cm}^2/\text{min} &= 64\pi r' \\ r' &= \frac{-1}{64\pi}\text{cm/min.} \end{aligned}$$

Thus when the radius is 8cm, the radius of the snowball is decreasing by $\frac{1}{64\pi}$ centimeters per minute.

(b) A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon, it is 45 feet above him. We want to know how fast is he distance between the boy and the balloon is increasing 3 seconds later.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.



Solution: We know one distance, and how fast it's changing. We want to know how fast another distance is changing, so the Pythagorean theorem, which relates distances to each other, seems like a reasonable choice. So we use the equation $d^2 = w^2 + h^2$.

We see that the height of the balloon is $h = 60$ft, and the derivative is $h' = 5$ft/s. The distance between the boy and the point under the balloon is $w = 45$ft and the derivative is $w' = 15$ft/s. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the current distance is 75 feet, and then that

$$\begin{aligned} 2dd' &= 2ww' + 2hh' \\ dd' &= ww' + hh' \\ 75\text{ft}d' &= 45\text{ft} \cdot 15\text{ft/s} + 60\text{ft} \cdot 5\text{ft/s} = 675\text{ft}^2/\text{s} + 300\text{ft}^2/\text{s} = 975\text{ft}^2/\text{s} \\ d' &= \frac{975}{75}\text{ft/s} = \frac{325}{25}\text{ft/s} = 13\text{ft/s.} \end{aligned}$$

Thus the distance between the boy and the balloon is increasing by 13 feet per second.