Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 8
Due Wednesday, October 23

This week's mastery quiz has three topics. Everyone should submit on M3. If you have a 2/2 on S5 or S6, you don't need to submit them. If you're unsure about your current grade, please check Blackboard.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Secondary Topic 5: Implicit Differentiation
- Secondary Topic 6: Related Rates

Name:

Recitation Section:

Major Topic 3: Optimization

(a) Classify the critical points and relative extrema of $g(x) = \dfrac{2x-1}{x^2+2}$.

Solution: We have

$$\begin{aligned} g'(x) &= \frac{2(x^2+2) - 2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2+2x+4}{(x^2+2)^2} \\ &= -2\frac{x^2-x-2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2} \end{aligned}$$

so the critical points are 2 and -1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$g'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$\begin{aligned} g''(x) &= -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4} \\ g''(-1) &= -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0 \\ g''(2) &= -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0. \end{aligned}$$

Thus $g''(-1) > 0$ so g has a minimum at -1; and $g''(2) < 0$ so g has a maximum at 2.

(b) The function $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ has absolute extrema either on the interval $(-1, 2)$, or on the interval $[-1, 2]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution: g is continuous on the closed interval $[-1, 2]$, so by the Extreme Value Theorem it has a maximum and a minimum on the interval. This must happen at a critical point or an endpoint. (This argument is necessary! Otherwise there's no reason to expect the largest local max to be a global max.)

We compute

$$g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x+1)(x-1)$$

which is zero at $0, 1, -1/2$. Then we compute

$$\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}$$

Thus g has an absolute maximum of 25 at 2, and an absolute minimum of 3 at 1.

Secondary Topic 5: Implicit Differentiation

(a) Find a formula for y' in terms of x and y if $xy^3 = \sqrt{x^2 + y^2}$.

Solution: Using implicit differentiation, we have

$$\begin{aligned} y^3 + 3xy^2y' &= \frac{2x + 2yy'}{2\sqrt{x^2 + y^2}} \\ &= \frac{x + yy'}{\sqrt{x^2 + y^2}} \\ y^3 - \frac{x}{\sqrt{x^2 + y^2}} &= \frac{yy'}{\sqrt{x^2 + y^2}} - 3xy^2y' \\ y' &= \frac{y^3 - \frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}} - 3xy^2} \\ &= \frac{y^3\sqrt{x^2 + y^2} - x}{y - 3xy^2\sqrt{x^2 + y^2}}. \end{aligned}$$

(b) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$\begin{aligned} 4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} \\ \frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx} \end{aligned}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \sqrt{5}(x - \sqrt{5}). \end{aligned}$$

Secondary Topic 6: Related Rates

A rocket is taking off with a perfectly vertical path, and is being tracked by a radar station on the ground four miles from the launch pad. We want to know how fast the rocket is rising when it is three miles high and its distance from the radar station is increasing at a rate of 3000 miles per hour.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.

Solution: We know one speed and want to know another, and we also know distances. This means we probably want to use the distance formula and take its derivative to find speeds.

We write $h = 3$mi, and can work out that $d = 5$mi. We know from the text of the problem that $d' =$ mi/h.

We know that $d^2 = h^2 + 4^2\text{mi}^2$ and thus $2dd' = 2hh'$. Plugging in values gives us

$$\begin{aligned} 2 \cdot 5\text{mi} \cdot 3000\text{mi/hr} &= 2 \cdot 3\text{mi} \cdot h' \\ h' &= 5000\text{mi/hr}. \end{aligned}$$

Thus the rocket is rising at 5000 miles per hour.