Math 1231 Fall 2024
Single-Variable Calculus I Section 11
Mastery Quiz 9
Due Wednesday, October 30

This week's mastery quiz has three topics. Everyone should submit on all three.

Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization
- Secondary Topic 7: Curve Sketching
- Secondary Topic 8: Physical Optimization

Name:

Recitation Section:

Major Topic 3: Optimization

(a) Find and classify the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

Solution: We have

$$f'(x) = \frac{1}{3}(x^3 - 3x)^{-2/3}(3x^2 - 3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

We could try the second derivative test, but we know it won't work at $\pm\sqrt{3}$ or at 0, since those are non-differentiable critical points. So we make a chart:

	$(x-1)$	$(x+1)$	$\sqrt[3]{x^2}$	$\sqrt[3]{(x^2-3)^2}$	f'
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$-$	$+$	$+$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$+$	$-$
$0 < x < 1$	$-$	$+$	$+$	$+$	$-$
$1 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	$+$
$\sqrt{3} < x$	$+$	$+$	$+$	$+$	$+$

So we see the function has a local maximum at -1 and a local minimum at 1; the critical points at $-\sqrt{3}, 0$, and $\sqrt{3}$ are neither minima nor maxima.

(b) The function $g(x) = x^3 - 3x^2 - 9x + 3$ has absolute extrema either on the interval $(-2, 4)$ or on the interval $[-2, 4]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution: g is continuous on the closed interval $[-2, 4]$ so by the extreme value theorem it has an absolute maximum and an absolute minimum.

We compute $g'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ is always defined, and is zero if $x = -1$ or $x = 3$. So the critical points are -1 and 3, and we need to check the points $-2, -1, 3, 4$.

$$g(-2) = 1 \qquad g(-1) = 8$$
$$g(3) = -24 \qquad g(4) = -17.$$

Thus g has a maximum of 8 at -1 and a minimum of -24 at 3.

Secondary Topic 7: Curve Sketching

Let $f(x) = \dfrac{(x-2)^2}{x-1}$. We can compute that

$$f'(x) = \frac{x(x-2)}{(x-1)^2}$$
$$f''(x) = \frac{2}{(x-1)^3}.$$

Sketch a graph of f. Your answer should discuss the domain, roots, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity and points of inflection.

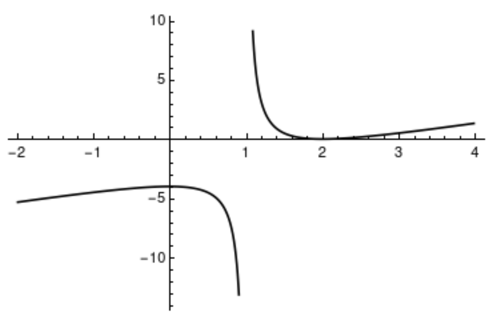
Solution: The function is defined for all real numbers except 1. We compute that $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$. There is a root of f at $x = 2$, and we compute that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

$f'(x)$ is undefined at $x = 1$ and is 0 at $0, 2$ so the critical points are $0, 1, 2$. We compute that $f(0) = -4$ and $f(2) = 0$. We make a chart:

	x	$x-2$	$(x-1)^{-2}$	$f'(x)$
$x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$1 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 0)$ and $(2, +\infty)$, and is decreasing on $(0, 2)$. It has relative a relative maximum at $(0, -4)$ and a relative minimum at $(2, 0)$; it doens't have a value at 1.

$f''(x)$ is undefined at 1 and is never 0, so the only possible point of inflection is 1. We see that $f''(x)$ is negative if $x < 1$ and positive if $x > 1$, so f is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$.



Secondary Topic 8: Physical Optimization

To check a bag on a certain airplane, the length plus width plus height must be less than or equal to 63in. Assuming the suitcase should be twice as long as it is wide, what height maximizes the volume of the suitcase? Justify your claim that this is a maximum.

Solution: Our objective function is $V = \ell wh$. We know that $\ell = 2w$ and that $\ell + w + h = 63$, thus that $3w + h = 63$ and so $h = 63 - 3w$. Then we have $V = 2w \cdot w \cdot (63 - 3w) = 126w^2 - 6w^3$.

$$V' = 252w - 18w^2 = 18w(14 - w)$$

so the critical points are $w = 0$ and $w = 14$.

We have three options for proving this is a maximum (we only need one):

(a) Extreme Value Theorem: The function $V(w) = 126w^2 - 6w^3$ is a continuous function, defined on the interval $[0, 21]$ (I'd also accept $[0, 63]$ here). Thus by the extreme value theorem there is an absolute maximum, which happens at a critical point or an endpoint. $V(0) = 0 = V(21)$ and $V(14) => 0$ so that value is a maximum.

(b) First Derivative Test: For $w < 14$ we have $V'(w) = 18w(14 - w) > 0$ so the function is increasing, and for $w > 14$ we have $V'(w) < 0$ so the function is decreasing. Thus we have a unique maximum at 14.

(c) Second derivative test: $V''(w) = 252 - 36w$. Then $V''(14) = 252 - 504 = -252 < 0$, which implies there is a relative maximum at 14. This doesn't really rigorously prove that this is an absolute maximum but I'll take it.

Finally, we asked about the *height* that maximizes the volume. This height is $h = 63 - 3w = 63 - 42 = 21$in.