Math 1231: Single-Variable Calculus 1
George Washington University   Fall 2024
Recitation 9

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October 29, 2024

Problem 1. Let $g(x) = x\tan(x)$. We want to sketch a graph of g.

1. What is the domain of g?
2. For simplicity, let's just look at $[-\pi/2, \pi/2]$. What can you say about any asymptotes it has?
3. Does this function have any roots you can find?
4. $g'(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. What are the critical points?
   (Hint: when is $\sin(x)\cos(x)$ positive and when is it negative?)
5. What are the critical values?
6. Where is g increasing and decreasing? Does it have maxima or minima?
7. $g''(x) = 2\sec^2(x)(1 + x\tan(x))$. Where are the potential points of inflection, and what are their values? Where is h concave up and down?
8. Sketch the graph.

Solution:

1. The domain of g is real numbers except $n\pi + \pi/2$.
2. In $[-\pi/2, \pi/2]$ we can't include the endpoints so we now have $(-\pi/2, \pi/2)$. $\lim_{x\to-\pi/2^+} g(x) = +\infty$ and $\lim_{x\to\pi/2^-} g(x) = +\infty$, which gives us asymptotes.

3. The function is 0 when $x = 0$ (and when $x = n\pi$ if we look farther out).

4. $g'(x) = \tan(x) + x\sec^2(x) = \frac{\sin(x)\cos(x)+x}{\cos^2(x)}$. It's not hard to see that when $-\pi/2 < x < 0$ then $g'(x) < 0$, and when $0 < x < \pi/2$ then $g'(x) > 0$, and $g'(0) = 0$. So the only critical point is at 0.

5. $g(0) = 0$.

6. And we saw that g is decreasing on $(-\pi/2, 0)$ and increasing on $(0, \pi/2)$. Thus g has a local minimum at 0. $g(0) = 0$.

7. $x \tan x \geq 0$ on $(-\pi/2, \pi/2)$, so $g''(x) \geq 0$ on $(-\pi/2, \pi/2)$, so the function is concave up everywhere.

8.

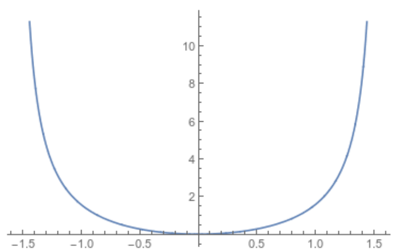

Figure 0.1: The graph of $g(x) = x\tan(x)$

Problem 2. Let $h(x) = \frac{x+2}{x-1}$. We want to sketch a graph of h.

1. What is the domain of h? What can you say about any asymptotes it has?

2. Does this function have any roots? Where?

3. What happens as x approaches $+\infty$? $-\infty$?

4. $h'(x) = -3(x-1)^{-2}$. What are the critical points and values?

5. Where is h increasing and decreasing? Does it have maxima or minima?

6. $h''(x) = 6(x-1)^{-3}$. Where are the potential points of inflection? Where is h concave up and down?

7. Sketch the graph.

Solution:

1. The domain of h is all real numbers except 1. We see that $\lim_{x\to 1^-} h(x) = -\infty$ and $\lim_{x\to 1^+} h(x) = +\infty$.

2. The function has a root at $x = -2$.

3. We have $\lim_{x\to+\infty} h(x) = \lim_{x\to-\infty} h(x) = 1$. (We can use L'Hôpital's rule or divide the top and bottom by x).

4. We have $h'(x) = \frac{(x-1)-(x+2)}{(x-1)^2} = -3(x-1)^{-2}$. This has no roots and fails to exist when $x = 1$. Thus there are no "real" critical points.

5. We make a chart for increase and decrease:

	-3	$(x-1)^{-2}$	$h'(x)$
$x < 1$	$-$	$+$	$-$
$1 < x$	$-$	$+$	$-$

   Thus h is decreasing everywhere. It has no local maxima or minima.

6. $h''(x) = 6(x-1)^{-3}$ is positive when $x > 1$ and negative when $x < 1$, so it is concave down on the left, and concave up on the right.

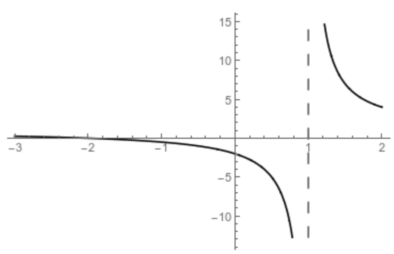

Figure 0.2: The graph of $h(x) = \frac{x+2}{x-1}$

Problem 3 (Bonus)**.** Let $g(x) = x^5 - 4x^3 + 4x + 7$. We want to sketch a graph of g.

1. What is the domain of g? What can you say about any asymptotes it has?

2. Does this function have any roots you can find?

3. What happens as x approaches $+\infty$? $-\infty$?

4. $g'(x) = 5x^4 - 12x^2 + 4$. What are the critical points?

   (Hint: if we set $u = x^2$ this becomes a quadratic, and we can factor it.)

5. What are the critical values?

   (Hint: $g(x) = 7 + x(x^4 - 4x^2 + 4) = 7 + x(u^2 - 4u + 4)$.)

6. Where is g increasing and decreasing? Does it have maxima or minima?

7. $g''(x) = 20x^3 - 24x$. Where are the potential points of inflection, and what are their values? Where is h concave up and down?

8. Sketch the graph.

Solution:

1. The domain of g is all reals. There are no asymptotes.

2. This function has one real root, but good luck finding it without a computer.

3. $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.

4. The derivative is defined everywhere. We can work out that if $u = x^2$ we have

   $$4x^5 - 12x^2 + 4 = 5u^2 - 12u + 4 = (5u-2)(u-2) = (5x^2-2)(x^2-2)$$

   so the derivative has four roots: where $x^2 = 2$ and where $x^2 = 2/5$. Thus the critical points are $x = \pm\sqrt{2}$ and $x = \pm\sqrt{2/5}$.

5.

$$\begin{aligned} g(-\sqrt{2}) &= 7 - \sqrt{2}(4 - 8 + 4) = 7 \\ g(\sqrt{2}) &= 7 + \sqrt{2}(4 - 8 + 4) = 7 \\ g(-\sqrt{2/5}) &= 7 - \sqrt{2/5}\left(\frac{4}{25} - \frac{8}{5} + 4\right) = 7 - \sqrt{2/5}\cdot\frac{64}{25} \\ g(\sqrt{2/5}) &= 7 + \sqrt{2/5}\left(\frac{4}{25} - \frac{8}{5} + 4\right) = 7 + \sqrt{2/5}\cdot\frac{64}{25}. \end{aligned}$$

6. We can make a chart:

	$5x^2-2$	x^2-2	g'
$x < -\sqrt{2}$	+	+	+
$-\sqrt{2} < x < -\sqrt{2/5}$	+	−	−
$-\sqrt{2/5} < x < \sqrt{2/5}$	−	−	+
$\sqrt{2/5} < x < \sqrt{2}$	+	−	−
$\sqrt{2} < x$	+	+	+

So f is increasing on $(-\infty, -\sqrt{2}) \cup (-\sqrt{2/5}, \sqrt{2/5}) \cup (\sqrt{2/5}, +\infty)$ and is decreasing on $(-\sqrt{2}, -\sqrt{2/5}) \cup (\sqrt{2/5}, \sqrt{2})$.

g has local maxima at $(-\sqrt{2}, 7)$ and at $\left(\sqrt{2/5}, 7 + \sqrt{2/5} \cdot \frac{64}{25}\right)$. It has local minima at $\left(-\sqrt{2/5}, 7 - \sqrt{2/5} \cdot \frac{64}{25}\right)$ and $(\sqrt{2}, 7)$.

7. $g''(x) = 4x(5x^2 - 6)$ so the possible points of inflection are 0 and $\pm\sqrt{6/5}$. We compute

$$\begin{aligned} g(-\sqrt{6/5}) &= 7 - \sqrt{6/5}\left(\frac{36}{25} - \frac{24}{5} + 4\right) = 7 - \sqrt{6/5} \cdot \frac{16}{25} \\ g(0) &= 7 \\ g(\sqrt{6/5}) &= 7 + \sqrt{6/5}\left(\frac{36}{25} - \frac{24}{5} + 4\right) = 7 + \sqrt{6/5} \cdot \frac{16}{25} \end{aligned}$$

To find concavity we can make another chart:

	$4x$	$5x^2 - 6$	g'
$x < -\sqrt{5/6}$	$-$	$+$	$-$
$-\sqrt{6/5} < x < 0$	$-$	$-$	$+$
$0 < x < \sqrt{6/5}$	$+$	$-$	$-$
$\sqrt{6/5} < x$	$+$	$+$	$+$

So g is concave upwards on $(-\sqrt{6/5}, 0) \cup (\sqrt{6/5}, +\infty)$, and concave downwards on $(-\infty, -\sqrt{6/5}) \cup (0, \sqrt{6/5})$. (So all the potential points of inflection are in fact points of inflection.)

8.

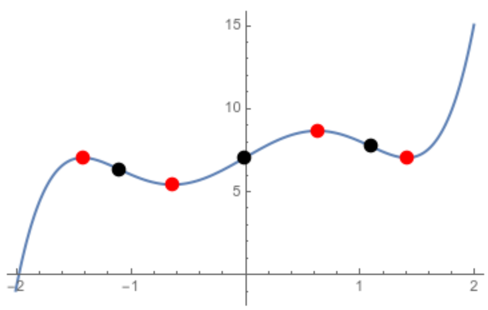

Figure 0.3: The graph of $g(x) = x^5 - 4x^3 + 4x + 7$. Critical points in red, and points of inflection in black.