Math 1232 Midterm Solutions

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- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has eight questions, over five pages. **You should not answer all eight questions.**
  - The first two problems are three pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
  - The remaining six problems represent topics S1 through S6. You will be graded on your **best three**, with a few possible bonus points if you also do well on the others.
  - Doing three secondary topics well is much better than doing five or six poorly.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

	a	b	c
M1			
M2			
S1		S2	
S3		S4	
S5		S6	
$\Sum$			/90

Problem 1 (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Compute $\frac{d}{dx}\sin(x)^{x^2+1}$.

Solution:

$$\begin{aligned} y &= \sin(x)^{x^2+1} \\ \ln(y) &= (x^2+1)\ln(\sin(x)) \\ \frac{y'}{y} &= 2x\ln(\sin(x)) + (x^2+1)\frac{\cos(x)}{\sin(x)} \\ y' &= y\left(2x\ln(\sin(x)) + (x^2+1)\frac{\cos(x)}{\sin(x)}\right) \\ &= \sin(x)^{x^2+1}\left(2x\ln(\sin(x)) + (x^2+1)\frac{\cos(x)}{\sin(x)}\right) \end{aligned}$$

(b) Compute $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$.

Solution: We can take $u = e^{2x}$ so $du = 2e^{2x}$. Then we have

$$\begin{aligned} \int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx &= \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin(u) + C. = \frac{1}{2}\arcsin(e^{2x}) + C. \end{aligned}$$

(c) Compute $\displaystyle\int_2^e \frac{3}{x\ln(x)^3}\,dx$.

Solution: We can use a u-substitution with $u = \ln(x)$, so $du = \frac{1}{x}\,dx$. Then we have

$$\begin{aligned} \int \frac{3}{x\ln(x)^3}\,dx &= \int \frac{3}{u^3}\,du = \frac{-3}{2u^2} + C = \frac{-3}{2\ln(x)^2} + C \\ \int_2^3 \frac{3}{x\ln(x)^3}\,dx &= \frac{-3}{2\ln(x)^2}\Big|_2^e = \frac{-3}{2} - \frac{-3}{2\ln(2)^2}. \end{aligned}$$

Alternatively, we can change the bounds of integration. We know $u(2) = \ln(2)$ and $u(e) = 1$, so we get

$$\begin{aligned} \int_2^e \frac{3}{x\ln(x)^3}\,dx &= \int_{\ln(2)}^1 \frac{3}{u^3}\,du = \frac{-3}{2u^2}\Big|_{\ln(2)}^1 \\ &= \frac{-3}{2} - \frac{-3}{2\ln(2)^2}. \end{aligned}$$

Problem 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) Compute $\int \frac{4+x}{(1+2x)(3-x)}\,dx =$

Solution:

$$\begin{aligned}
\frac{4+x}{(1+2x)(3-x)} &= \frac{A}{1+2x} + \frac{B}{3-x} \\
4+x &= A(3-x) + B(1+2x) \\
7 &= 7B \Rightarrow B = 1 \\
4 &= 3A + 1 \Rightarrow A = 1 \\
\int \frac{4+x}{(1+2x)(3-x)} &= \int \frac{1}{3-x} + \frac{1}{1+2x}\,dx \\
&= -\ln|3-x| + \frac{1}{2}\ln|1+2x| + C.
\end{aligned}$$

(b) $\int \tan^4(3x)\,dx.$

Solution:

$$\begin{aligned}
\int \tan^4(3x)\,dx &= \int (\sec^2(3x)-1)\tan^2(3x)\,dx = \int \sec^2(3x)\tan^2(3x) - \tan^2(3x)\,dx \\
&= \int \sec^2(3x)\tan^2(3x) - \big(\sec^2(3x)-1\big)\,dx = \int \sec^2(3x)\tan^2(3x) - \sec^2(3x) + 1\,dx \\
&= \frac{1}{9}\tan^3(3x) - \frac{1}{3}\tan(3x) + x + C. \\
&\left(= \frac{1}{9}\sec^2(3x)\tan(3x) - \frac{4}{9}\tan(3x) + x + C.\right)
\end{aligned}$$

(c) $\int x^2\sin(3x)\,dx =$

Solution:

$$\begin{aligned}
\int x^2\sin(3x)\,dx &= \frac{-1}{3}\cos(3x)x^2 - \int \frac{-1}{3}\cos(3x)\cdot 2x\,dx \\
&= \frac{-x^2}{3}\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx \\
&= \frac{-x^2}{3}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}\sin(3x)x - \int \frac{1}{3}\sin(3x)\,dx\right) \\
&= \frac{-x^2}{3}\cos(3x) + \frac{2x}{9}\sin(3x) + \frac{2}{27}\cos(3x) + C.
\end{aligned}$$

Problem 3 (S1)**.**
Let $h(x) = \sqrt{5x^3+3x+1}$. Compute $(h^{-1})'(3)$.

Solution: By the Inverse Function Theorem, we know that

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))}.$$

Guess and check shows that $h(1) = 3$ so $h^{-1}(3) = 1$. And we know that

$$h'(x) = \frac{1}{2}(5x^3+3x+1)^{-1/2}(15x^2+3)$$

and thus

$$h'(2) = \frac{1}{6}(18) = 3.$$

Thus

$$(h^{-1})'(3) = \frac{1}{3}.$$

Problem 4 (S2)**.** Find $\lim_{x\to+\infty} x^{1/\ln(1\,+\,2x)}$.

Solution: We can take

$$\begin{aligned}
y &= x^{1/\ln(1\,+\,2x)} \\
\ln(y) &= \frac{1}{\ln(1+2x)}\ln(x) \\
\lim_{x\to+\infty}\ln(y) &= \lim_{x\to+\infty}\frac{\ln(x)^{\nearrow\infty}}{\ln(1+2x)_{\searrow\infty}} \\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{1/x}{2/(1+2x)} = \lim_{x\to+\infty}\frac{1+2x^{\nearrow\infty}}{2x_{\searrow\infty}} \\
&=^{\text{L'H}} \lim_{x\to+\infty}\frac{2}{2} = 1. \\
\lim_{x\to+\infty} y &= e.
\end{aligned}$$

Problem 5 (S3)**.** How many intervals do you need with the trapezoid rule to approximate $\int_{-2}^{4} x^2\,dx$ to within 1? Compute that approximation.

Solution: We know that $f''(x) = 2$, so $K = 2$, and

$$\begin{aligned}
E &\leq \frac{K(4+2)^3}{12\cdot n^2} \leq 1 \\
2\cdot 6^3 &\leq 12\cdot n^2 \\
6^2 &\leq n^2
\end{aligned}$$

so we need to take $n = 6$ and use 6 intervals. Then

$$\begin{aligned}
\int_{-2}^{4} f(x)\,dx &\approx \frac{1}{2}\left(f(-2)+2f(-1)+2f(0)+2f(1)+2f(2)+2f(3)+f(4)\right) \\
&\approx \frac{1}{2}\left(4+2+0+2+8+18+16\right) = 25.
\end{aligned}$$

Problem 6 (S4)**.** $\displaystyle\int_{-\infty}^{1}\frac{3}{x^2+1}\,dx$

Solution:

$$\begin{aligned}
\int_{-\infty}^{1}\frac{3}{x^2+1}\,dx &= \lim_{t\to-\infty}\int_{t}^{1}\frac{3}{x^2+1}\,dx \\
&= \lim_{t\to-\infty} 3\arctan(x)\Big|_t^1 = \lim_{t\to-\infty} 3\arctan(1) - 3\arctan(t) \\
&= \frac{3\pi}{4} - \lim_{t\to-\infty} 3\arctan(t) = \frac{3\pi}{4} - \frac{-3\pi}{2} = \frac{9\pi}{4}.
\end{aligned}$$

Problem 7 (S5)**.** Let $f(x) = \frac{1}{3}(x^2-2)^{3/2}$ Find the arc length of the graph of f for x between 2 and 3.

Solution: We compute $f'(x) = \frac{1}{2}(x^2-2)^{1/2} \cdot 2x = x\sqrt{x^2-2}$. So the arc length is

$$L = \int_2^3 \sqrt{1 + x^2(x^2-2)}\,dx = \int_2^3 \sqrt{1 - 2x^2 + x^4}\,dx$$
$$= \int_2^3 x^2 - 1\,dx = \frac{x^3}{3} - x\Big|_2^3 = (9-3) - (8/3-2) = 8 - 8/3 = 16/3.$$

Problem 8 (S6)**.** Find the (specific) solution to $y' = \frac{3y^2}{x}$ if $y(1) = 1$.

Solution:

$$\frac{dy}{dx} = \frac{3y^2}{x}$$
$$\frac{dy}{3y^2} = \frac{1}{x}\,dx$$
$$\int \frac{dy}{3y^2} = \int \frac{1}{x}\,dx$$
$$\frac{-1}{3y} = \ln|x| + C$$
$$3y = \frac{-1}{\ln|x| + C}$$
$$y = \frac{-1}{3\ln|x| + C}.$$

Plugging in $x = 1, y = 1$ gives

$$1 = \frac{-1}{3 \cdot 0 + C} = \frac{-1}{C}$$
$$C = -1$$
$$y = \frac{-1}{3\ln|x| - 1}.$$