4 Sequences and Series

In this section we're going to introduce a completely new set of ideas. Well, sort of.

So far in calculus we've studied *continuous* things: functions, derivatives, and integrals. What these all have in common is that you can chop them into pieces as small as you want. In this section we'll see what happens if we study *continuous* things, that genuinely have a smallest possible size.

Conceptually, thinking about discrete things is easier: there are fewer things that can happen, so everything is conceptually simpler. And in a real sense, the world as we perceive it is basically discrete, since you're never going to make infinitely many measurements.

So then... why did we start with the continuous version? It's conceptually harder and more artificial, but it is far, far easier to do *computations* in the continuous realm. So the way we mostly solve real problems is to find a way to pretend our discrete question is really continuous, and then solve the continuous question, and hope we can use that to answer our original discrete question. We'll see that throughout this section as well.

4.1 Sequences

Definition 4.1. A *sequence* of real numbers is a (usually infinite) ordered list of real numbers. We write $(a_n)_{n=1}^{\infty}$ for the sequence

$$(a_1, a_2, a_3, \dots)$$

where each a_n is a real number.

We can think of a sequence as the discrete equivalent of a function. In particular, a sequence is a function from the natural numbers to the real numbers, where $f(n)$ is the nth element of the sequence. Thus it's a function that only allows integer inputs, unlike continuous functions that allow any real number as an input.

Example 4.2. A few examples of sequences. Some of these will look familiar:

(a) $(1, 1, 1, 1, 1, \dots)$

(b) $(1, 2, 3, 4, \dots)$

(c) $(2^{10}, 17, \sqrt[5814]{3^{11} - 1}, 1, 1, 1, \dots)$

(d) $(1, 1, 2, 3, 5, 8, 13, \dots)$

(e) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

(f) $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$

(g) $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots)$

(h) $(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, \dots)$

In most of these sequences the pattern is pretty obvious. In sequence (a) we have $a_n = 1$. In sequence (b) we have $a_n = n$ and in sequence (e) we have $a_n = 1/n$. Less obviously, in sequence (g) we have $\frac{n}{n+1}$ and in (h) we have $\cos(n\pi/6)$.

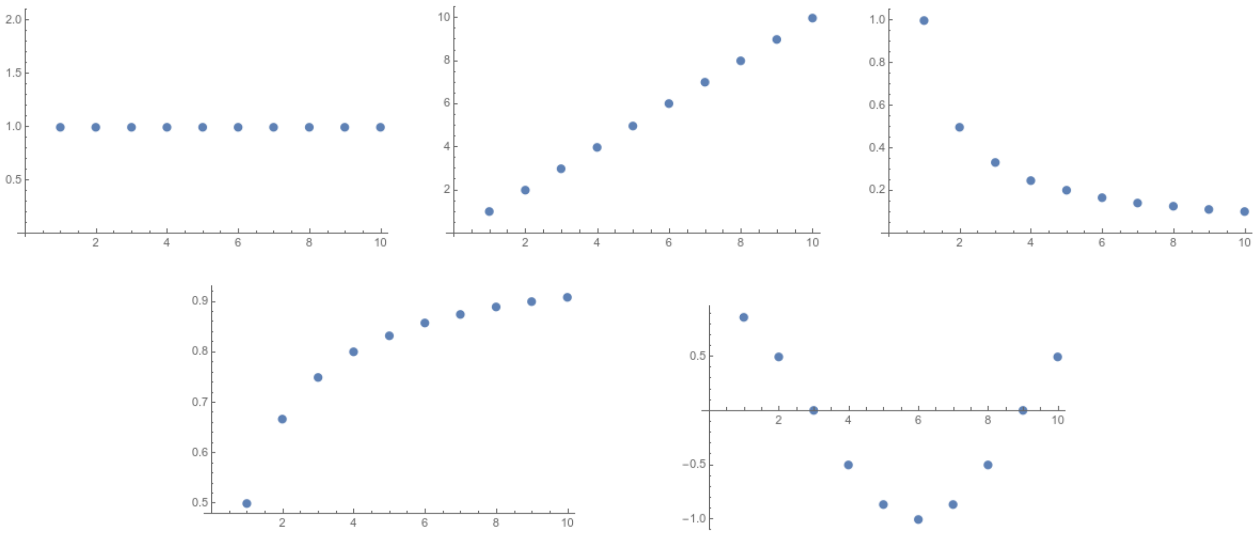

Figure 4.1: The graphs of (a), (b), (e), (g), (h)

However, not all sequences have nice descriptions like this. Sequence (d) is the *fibonacci sequence*, which is defined "inductively" or "recursively" by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. (This sequence was originally defined to work on problems about rabbit-breeding; it appears often in nature).

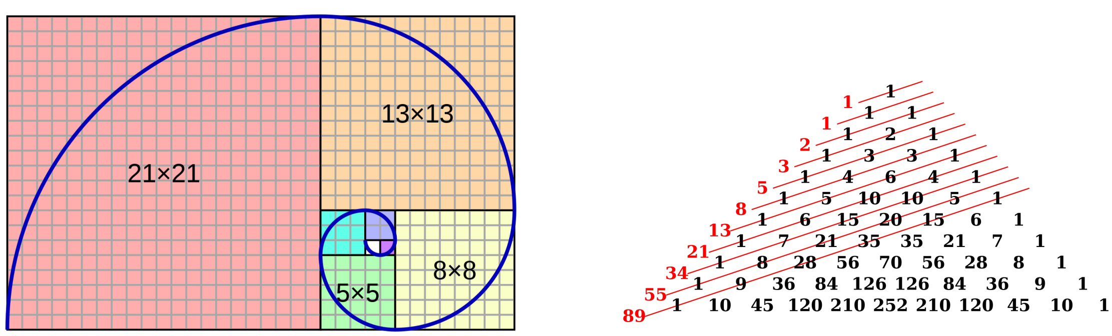

Figure 4.2: The Fibonacci sequence is related to the Golden Ratio and to Pascal's Triangle. Left: Jahobr, CC0; Right: RDBury, CC BY-SA 3.0, both via Wikimedia Commons

Even worse are sequences like (c) which show no particular pattern at all; these are still sequences.

Example 4.3. What is the general form of the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots)$? We see that $a_n = \frac{1}{n^2}$.

What about $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots)$? $a_n = \frac{1}{2^n}$.

We can see that some of these sequences look like they're "going somewhere"—in fact, sequence (a) is already there! But sequences (e), (f), and (g) all seem to be getting closer and closer to some value.

When the terms of a sequence are getting closer and closer to some value, we say that it has a *limit*. In particular, we say the sequence (a_n) has a limit L in the real numbers if we can make the numbers a_n get as close to L as we want just by taking n to be sufficiently big.

Example 4.4. The sequence $(1/n^2)$ has a limit of 0.

The sequence $1/2^n$ also has a limit of 0.

Just like with functions, we can use ϵ to make this definition more rigorous:

Definition 4.5. Let (a_n) be a sequence of real numbers. We say that (a_n) has a limit L, and write $\lim_{n\to+\infty} a_n = L$, if, for every real number $\epsilon > 0$, there is a natural number N such that, whenever $n \geq N$, $|a_n - L| < \epsilon$.

If a sequence has a limit in the real numbers we say the sequence *converges*. Otherwise we say the sequence *diverges*, and the limit *does not exist*.

Example 4.6. Prove that $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Fix some $\epsilon > 0$. Then let $N > 1/\epsilon$. If $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and thus $|a_n - 0| < \epsilon$. So by definition, $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Example 4.7. Prove that $\lim_{n\to+\infty}(-1)^n$ does not exist.

Heuristically, we notice that this sequence "bounces around"; it doesn't get closer to just one value. Informally, the sequence has two different values it reaches infinitely often, so it doesn't have one single limit. But we can also make this rigorous with an $\epsilon - N$ argument:

For a limit to exist, a certain statement needs to be true for any positive real number. So to prove that a limit does not exist we just need to find *one* real number for which the statement is false.

So let $\epsilon = 1$, and suppose a limit L exists. Then we can find a N such that if $n \geq N$, then $|(-1)^n - L| < 1$. In particular, we can find both even n and odd n, and so it must be the case that $|1 - L| < 1$ and $|-1 - L| < 1$. But there is no number L that makes this true. So no limit exists.

Computing limits in this way is important, and a good exercise, but a bit painful. And just like with functions, we have limit laws that make the process much easier.

Proposition 4.8. *If (a_n) and (b_n) are convergent sequences and c is a constant, then*

- $\lim\limits_{n\to+\infty} a_n \pm b_n = \lim\limits_{n\to+\infty} a_n \pm \lim\limits_{n\to+\infty} b_n$
- $\lim_{n\to+\infty} c = c$
- $\lim_{n\to+\infty} ca_n = c\lim_{n\to+\infty} a_n$
- $\lim_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} a_n \lim_{n\to+\infty} b_n$
- $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n}$ *if* $\lim_{n\to+\infty} b_n \neq 0.$
- $\lim_{n\to+\infty} a_n^p = (\lim_{n\to+\infty} a_n)^p$ *if* $p, a_n > 0.$

Example 4.9. What is $\lim_{n\to+\infty} \frac{n+1}{n}$?

We can write

$$\lim_{n\to+\infty} \frac{n+1}{n} = \lim_{n\to+\infty}(1+\frac{1}{n})$$
$$= \lim_{n\to+\infty} 1 + \lim_{n\to+\infty} \frac{1}{n} = 1+0 = 1.$$

Example 4.10 (recitation)**.** What is the limit of the sequence $\sqrt{n+1} - \sqrt{n}$?

Using a familiar trick from Calculus 1, we see

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$
$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$
$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Thus

$$\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

That last step was arguably a bit fuzzy. There are a few ways to make it rigorous; one is to argue that our sequence "looks like" $\frac{1}{\sqrt{n}}$. In particular, our sequence is smaller than $\frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} \to 0$, so our sequence should also get close to zero. We can make that precise with the Squeeze Theorem:

Theorem 4.11 (Squeeze Theorem)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = L$ then $\lim_{n\to+\infty} b_n = L$.*

To continue the earlier example, we have

$$\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$
$$0 \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

We know that $\lim_{n\to+\infty} 0 = 0$. And we can work out that

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{\sqrt{n}} &= \lim_{n\to+\infty} \left(\frac{1}{n}\right)^{1/2} \\ &= \left(\lim_{n\to+\infty} \frac{1}{n}\right)^{1/2} \\ &= 0^{1/2} = 0.\end{aligned}$$

Then by the squeeze theorem, $\lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

Example 4.12. What is $\lim_{n\to+\infty} \frac{\sin n}{n}$?

This is a classic use case for the Squeeze Theorem. We know that $-1 \leq \sin n \leq 1$ for any n. So $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ for any n. We know that $\lim_{n\to+\infty} \frac{1}{n} = 0$, and similarly $\lim_{n\to+\infty} \frac{-1}{n} = -\lim_{n\to+\infty} \frac{1}{n} = 0$. So by the squeeze theorem, $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$.

But ultimately we can replace a lot of this with all the work we did with functions—since a sequence is secretly just a function anyway.

Theorem 4.13. *Suppose $f(x)$ is a function such that $f(n) = a_n$ for every natural number n, and $\lim_{x\to+\infty} f(x) = L$. Then $\lim_{n\to+\infty} a_n = L$.*

If $\lim_{n\to+\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to+\infty} f(a_n) = f(L).$$

Example 4.14. What is $\lim_{n\to+\infty} \frac{n}{n+1}$? We see that if $f(x) = \frac{x}{x+1}$, then $f(n) = a_n$, so we can compute

$$\lim_{n\to+\infty} \frac{n}{n+1} = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x/x}{(x+1)/x} = \lim_{x\to+\infty} \frac{1}{1+1/x} = 1.$$

Thus $\lim_{n\to+\infty} \frac{n}{n+1} = 1$.

Example 4.15. What is $\lim_{n\to+\infty} \frac{\ln n}{n}$?

We write $f(x) = \frac{\ln x}{x}$, and then $f(n) = a_n$. By L'Hôpital's rule, we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Thus we also have that $\lim_{n\to+\infty} \frac{\ln n}{n} = 0$.

However, this only works in one direction! If the function limit exists, then the sequence limit exists. But the converse is not true.

Example 4.16. What is $\lim_{n\to+\infty} \sin(n\pi)$?

Naively, we might argue this: Let $g(x) = \sin(x\pi)$. Then $\lim_{x\to+\infty} g(x)$ does not exist, since the function varies between -1 and 1 no matter how large we let x grow. Thus the limit does not exist.

However, our theorem only applies when $\lim_{x\to+\infty} g(x)$ exists; it tells us nothing if the limit of our function does not converge. In fact, for every n we have $\sin(n\pi) = 0$, and thus

$$\lim_{n\to+\infty} \sin(n\pi) = \lim_{n\to+\infty} 0 = 0.$$

But the real limitation is: not every sequence can be expressed reasonably as a function of the real numbers at all.

Definition 4.17. If n is a natural number, we define n *factorial*, written $n!$, to be

$$n! = n \cdot (n-1) \dots 2 \cdot 1.$$

This is the product of all positive integers less than or equal to n.

These will come up a lot in the remainder of this course.

Example 4.18. What is $\lim_{n\to+\infty} \frac{n!}{n^n}$?

We calculate that

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{n \cdot n \cdot n \dots n \cdot n} = \frac{1}{n} \cdot \frac{n(n-1)(n-2)\dots(2)}{n^{n-1}}.$$

It's clear that the large fraction is between 0 and 1 since the numerator is positive, but smaller than the denominator. Thus we have $0 \leq a_n \leq \frac{1}{n}$, and $\lim_{n\to+\infty} 0 = \lim_{n\to+\infty} \frac{1}{n} = 0$. By the squeeze theorem, $\lim_{n\to+\infty} a_n = 0$.

And just like with functions, we sometimes have sequences wtih infinite limits.

Example 4.19. $\lim_{n\to+\infty} n = +\infty$.

$\lim_{n\to+\infty} -n^2 = -\infty$.

4.1.1 Completeness

There's one important note I want to make here about the way sequences work, and the importance of the real numbers.

We would like to say that every sequence either goes to infinity or has a (finite) limit. Unfortunately, this isn't the case, because a sequence can bounce up and down without ever settling on one value (remember $(-1)^n$). But if a sequence doesn't "bounce around" then we know it must either have a limit or go to infinity.

Definition 4.20. A sequence is *(monotonically) increasing* if $a_{n+1} \geq a_n$ for all n. A sequence is *(monotonically) decreasing* if $a_{n+1} \leq a_n$ for all n. In either case we say that such a sequence is *monotonic.*

A sequence is *bounded above* if there is an A such that $a_n \leq A$ for all n. A sequence is *bounded below* if there is an A such that $a_n \geq A$ for all n. A sequence that is bounded above and bounded below is *bounded.*

A monotone sequence doesn't bounce around; a bounded sequence doesn't go to infinity. In the real numbers, a sequence with both of these properties must have a limit.

Fact 4.21. *Every increasing sequence of real numbers that is bounded above converges to some real number. Every decreasing sequence of real numbers that is bounded below converges to some real number. In particular, every bounded monotonic sequence is convergent.*

Remark 4.22. The idea here is that every sequence that "should" have a finite limit does. If the terms get closer to each other, there is some limit they approach.

Example 4.23. $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$.

If $0 \leq x \leq 2$ then $x \leq \sqrt{2x} \leq 2$. Thus since the first element is between 0 and 2, the sequence is increasing, and every element is ≤ 2, so the sequence is bounded above by 2. Thus it must converge.

Can we see what it must converge to? If we look at the sequence $a_n^2/2$ we have $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \dots$ and get the same sequence again, just "shifted by one." So

$$L = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{a_n^2}{2} = \frac{\left(\lim_{n\to+\infty} a_n\right)^2}{2} = \frac{L^2}{2}.$$

Thus $2L = L^2$ and $L = 2$.

Alternatively we can notice that $a_n = 2^{1-\frac{1}{2^n}}$. Then

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} 2^{1-\frac{1}{2^n}} = 2^{\left(\lim_{n\to+\infty} 1-\frac{1}{2^n}\right)} = 2^{1-0} = 2.$$

4.2 Series

In this section we will discuss a particular type of sequence called a series. Series are powerful and flexible tools that show up in many places in mathematics; they are used to compute approximations, they underlie integrals, and they are often used to solve differential equations.

But at base, we can think of a series as a sort of a discrete version of the integral. The integral is "continuous", which means it adds up values from every point in the domain; a series will add up the values at only distinct, separated points in the domain.

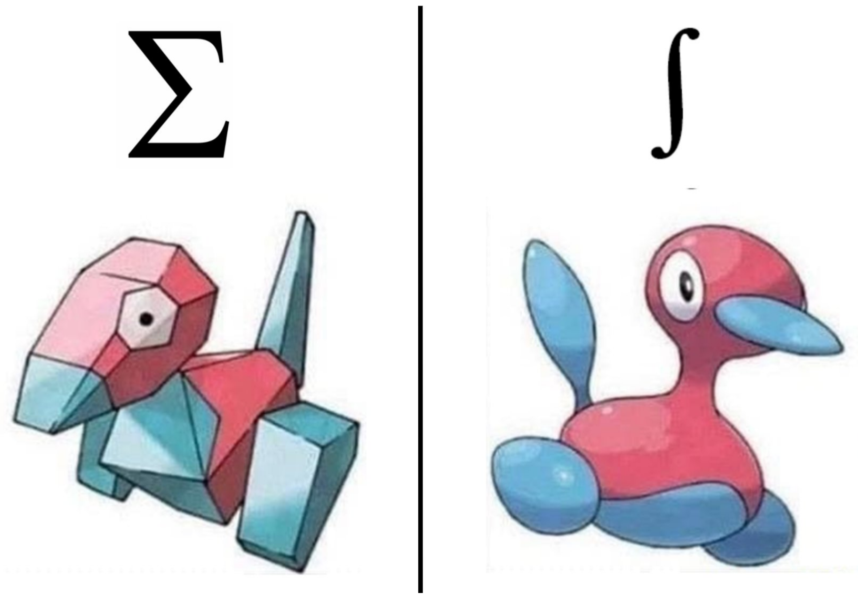

Figure 4.3: Meme courtesy of `@howie_hua` on Twitter

Definition 4.24. A *series* is a "sequence of partial sums." That is, a series is a sequence $(s_n)_{n=1}^{+\infty}$ where for some other real sequence (a_n) we have

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

If the sequence (s_n) is convergent and $\lim_{n\to+\infty} s_n = s$, then we say the series $\sum a_n$ converges to s, which is the *sum* of the series. We write

$$\sum_{n=1}^{\infty} a_n = s \qquad \text{or} \qquad a_1 + a_2 + \cdots + a_n + \cdots = s.$$

If (s_n) is divergent, then the series is also divergent.

Example 4.25. A couple of the sequences we saw in the last section are "really" series.

- $1, 2, 3, \dots$ can be viewed as $\sum_{i=1}^{\infty} 1$.
- Any infinite decimal representation is really a series: we have

$$\pi = 3 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + \dots$$

Example 4.26. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$ is the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. We see that the partial sum $s_n = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$, and thus $\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} 1 - \frac{1}{2^n} = 1 - 0 = 1$.

Remark 4.27. Notice that if the terms of a series are non-negative, then the sequence of partial sums is monotone increasing. Thus a series of positive terms either converges, or goes to infinity.

Example 4.28. The series $\sum_{n=1}^{\infty} (-1)^n$ has a sequence of partial sums $(-1, 0, -1, 0, \dots)$ and thus neither converges nor goes to infinity. But the terms are not all non-negative.

4.2.1 Telescoping Series and the Fundamental Theorem of Calculus

Series are the discrete version of integrals, but in general they're much harder to exactly compute. This is because we don't really have the Fundamental Theorem of Calculus—or at least, not in a useful way.

It's maybe worth thinking for a minute about what a discrete *derivative* would look like. In the continuous case, we say that the derivative approximates $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. Even more informally, we say that $f'(x)$ is roughly the amount f increases if you increase x by one. But that's not quite right, because we're actually taking a limit as Δx gets very small, and so Δx can be much smaller than 1.

But in our discrete case, you can't have steps smaller than one. So the equivalent of the derivative would be $\dfrac{a_{n+1} - a_n}{(n+1) - (n)} = a_{n+1} - a_n$. This "difference quotient" is a perfectly useful calculation that shows up in a lot of contexts, but we won't talk about it much more in this course.

If we want to use the Fundamental Theorem of Calculus, we'd need to find a way to write the term inside our sum as a difference of two consecutive terms of a series. This is always technically possible, since your series itself is a sequence with the right differences of terms. But it's only rarely possible to view your terms as the difference quotients of a useful series.

Example 4.29. What is $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$?

Our sequence looks like

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$$

which looks like it converges. By doing a partial fraction decomposition, we can write $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Then our partial sums are

$$\begin{aligned} s_n &= \sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &= 1 - \frac{1}{n}. \end{aligned}$$

Thus $\sum_{i=2}^{n} \frac{1}{n^2-n} = \lim_{n\to+\infty} 1 - \frac{1}{n} = 1$.

A series that works like this is called a *telescoping series*.

Example 4.30. Consider the series $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$. We can look at this as

$$\sum_{n=1}^{\infty} \log(n+1) - \log(n).$$

Then we can observe

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \log(n+1) - \log(n) \\ &= \big(\log(k+1) - \log(k)\big) + \big(\log(k) - \log(k-1)\big) + \dots + \big(\log(3) - \log(2)\big) + \big(\log(2) - \log(1)\big) \\ &= \log(k+1) - \log(1). \end{aligned}$$

$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \log(k+1) = \infty.$

Thus this sum diverges.

4.2.2 Series Rules

Just like with integrals, we can add series easily, and we can do scalar multiplication to them.

Proposition 4.31. *If $\sum a_n$ and $\sum b_n$ are convergent series, then*

- $\sum c a_n = c \sum a_n$.
- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$.

And just like with integrals, we technically can multiply series together, but it's complicated and hard to use:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

This operation is sometimes referred to as *convolution*. It it is too complicated to be terribly useful to us right now, but it often comes up in signal processing and more sophisticated approaches to differential equations.

4.2.3 Geometric Series

There's one more type of series that we can actually compute, which winds up being really important. These series don't *actually* telescope, but we can easily turn them into something that does.

Definition 4.32. A *geometric series* is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

for some real numbers a and r.

Some people prefer to think of a geometric series as $\sum_{n=1}^{\infty} ar^n$. I'm one of them, actually, but your textbook isn't. It doesn't really matter which convention you use as long as you're consistent.

Can we add these series up? Let's cheat: we'll assume it's possible, and figure out what the sum should be. So let's start out assuming that $\sum_{n=1}^{\infty} ar^{n-1}$ converges to some number L. Then we have

$$\begin{aligned}
rL &= \sum_{n=1}^{\infty} ar^n = ar + ar^2 + r^3 + \dots \\
&= (-a) + \left(a + ar + ar^2 + ar^3 + \dots\right) = -a + \sum_{n=1}^{\infty} ar^{n-1} \\
&= -a + L \\
(r-1)L &= -a \\
L &= \frac{a}{1-r}.
\end{aligned}$$

For some questions, this answer is fine. We already argued in example 4.26 that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. This is a geometric series with $a = r = \frac{1}{2}$, and thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1.$$

However, if we take this argument literally and don't do any more work, it suggests that $\sum_{n=1}^{\infty} 2^{n-1} = \frac{1}{1-2} = -1$, which is clearly absurd. (Well, usually. There's a trick called "regularization" that physicists use this for). But the basic idea is sound. First, we've shown that *if* the sume converges, it has to converge to $\frac{a}{1-r}$. And second, we can make the same argument a bit more carefully, paying attention to the limit, and getting something that actually works.

Let $s_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Then

$$\begin{aligned} rs_n &= \sum_{i=1}^{n} ar^i = ar + ar^2 + \cdots + ar^n \\ &= s_n - a + ar^n \\ (r-1)s_n &= a(r^n - 1) \\ s_n &= a\frac{r^n - 1}{r-1}. \end{aligned}$$

We can think of this as a sort of anti-difference quotient: we have a closed-form formula for the nth partial sum.

If we take the limit as n goes to infinity, this diverges if $|r| \geq 1$. If $|r| < 1$, it converges, and we get the formula $\lim_{n \to +\infty} s_n = \frac{a}{1-r}$. We summarize this result:

Proposition 4.33. *If* $\sum_{n=1}^{\infty} ar^{n-1}$ *is a geometric series and* $|r| < 1$*, then*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ *then the series diverges.*

Example 4.34. What is $\sum_{n=1}^{\infty} \frac{2}{3^n}$? This is a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3}$. (Note that $a = 2/3$ because a is the first term of the series.) So

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2/3}{1 - 1/3} = 1.$$

Example 4.35. What is $\sum_{n=1}^{\infty} \frac{2^n}{3}$? This is a geometric series, this time with $a = 2/3$ and $r = 2$. Since $|2| \geq 1$ the series diverges.

We can also use this technique to turn infinite repeating decimals into integer fractions.

Example 4.36. Can we write $4.\overline{13}$ as a ratio of integers?

We have

$$4.\overline{13} = 4 + \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \ldots$$

After the first term we have a geometric series with $a = \frac{13}{100}$ and $r = \frac{1}{100}$, so the sum is

$$\frac{a}{1-r} = \frac{13/100}{99/100} = \frac{13}{99}.$$

Thus

$$4.\overline{13} = 4 + \frac{13}{99} = \frac{409}{99}.$$

Example 4.37. Does $\sum_{n=1}^{\infty} 3^{2n} 2^{2-3n}$ converge or diverge?

This series looks like $\frac{9}{2} + \frac{3^4}{2^4} + \frac{3^6}{2^7} + \dots$. This is a geometric series with $a = \frac{9}{2}$ and $r = \frac{9}{8}$. Thus $|r| > 1$ and so the series diverges.

4.2.4 The Harmonic Series

There's one more series we can look at before we start building a general theory. This may be the single most important specific example we have.

Example 4.38. One of the most important series is the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$. (It underlies among other things the Riemann zeta function which controls the distribution of prime numbers). Does it converge or diverge?

There's no really generalizable argument that applies here. But if $s_n = \sum_{i=1}^{n} \frac{1}{n}$ is the sequence of partial sums, then

$$\begin{aligned}
s_1 &= 1 > \frac{1}{2} \\
s_2 &= 1 + \frac{1}{2} > 2 \cdot \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 3 \cdot \frac{1}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 4 \cdot \frac{1}{2}.
\end{aligned}$$

In particular, we see that $s_{2^{n-1}} > \frac{n}{2}$, and thus the sequence of partial sums increases without bound, and diverges to $+\infty$.

Remark 4.39. We will see that in some sense, the harmonic series is as small as it can get and still diverge.

Example 4.40 (Bonus Example)**.** The Kempner Series is the harmonic series, except we leave out every term where a 9 appears in the denominator. We claim that this series converges. (Yes, seriously. See also `http://www.smbc-comics.com/index.php?id=3777`).

We divide the series up according to the number of digits in the denominator. Among denominators with k digits, there are at most $8 \cdot 9^{k-1}$ since there are eight possibilities for the first digit (which cannot be 0 or 9) and 9 possibilities for the other digits (which cannot be 9). And each number is at least 10^{k-1}, so each term with k digits in the denominator is at most 10^{1-k}.

Then if we sum up all the terms with k digits in the denominator, we have $8 \cdot 9^{k-1}$ terms each of which is at most 10^{1-k} and so our sum is at most $\frac{8 \cdot 9^{k-1}}{10^{k-1}}$.

Now if we sum up the whole series, that's the same as summing up each set of k-digit denominators, and then summing all those sums. So we have

$$K \leq \sum_{k=1}^{\infty} 8 \frac{9^{k-1}}{10^{k-1}} = \sum_{k=1}^{\infty} 8 \left(\frac{9}{10} \right)^{k-1}.$$

This right-hand sum should look familiar; it's a geometric series. We have $a = 8$ and $r = \frac{9}{10}$, so the sum is

$$K \leq \sum_{k=1}^{\infty} 8 \left(\frac{9}{10} \right)^{k-1} = \frac{8}{1 - 9/10} = 80.$$

(A.J. Kempner first studied this series in 1914, and came up with the above argument. In 1979 Robert Baille showed that $K \approx 22.9$.)

Remark 4.41. In fact, if you take the harmonic series pick *any* string of digits, and remove terms with that string in the denominator, you get a convergent series, for basically the same reason.

4.3 The Divergence and Integral Tests

Now we can start building some general theoretical tools for understanding whether series converge.

4.3.1 The Divergence test

In the last section, we showed the harmonic series diverged by showing it was bigger than an infinite sum of a constant. This makes sense, because if you add the same number to itself infinitely many times, you will never get a finite amount. In fact, series can only converge if the terms get increasingly small as you go further into the series.

Proposition 4.42. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n \to +\infty} a_n = 0$*. Thus if* $\lim_{n \to +\infty} a_n \neq 0$*, or if the limit does not exist, then* $\sum_{n=1}^{\infty} a_n$ *does not converge.*

Remark 4.43. The converse is not true. The divergence test can be used to show a series diverges; it cannot show that a series converges.

The divergence test winds up being a sort of first-pass filter. It lets us check that a series diverges really quickly, but can never tell us that a series converges.

Example 4.44. Consider the series $\sum_{n=1}^{\infty} 1$. We can see that $\lim_{n\to+\infty} 1 = 1 \neq 0$, so this series diverges. (We can see that in other ways by seeing that it must go to ∞.)

Example 4.45. Consider the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$. We can see that $\lim_{n\to+\infty} \frac{n}{n+1} = 1 \neq 0$. Thus this series diverges.

Example 4.46. Consider the series $\sum_{n=1}^{\infty}(-1)^n$. We can compute $\lim_{n\to\infty}(-1)^n$, but this limit does not exist. Thus by the divergence test, the series diverges (as we saw in example 4.28).

Example 4.47. The divergence test tells us nothing about the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to+\infty} \frac{1}{n} = 0$, so we have no information. But we know that the harmonic series diverges by the argument in example 4.38.

This is a good example of how the divergence test can't show us a series converges. The harmonic series "passes" the divergence test: the terms go to zero. But that doesn't mean the series converges, and in fact it does not.

4.3.2 The Integral Test

So how can we tell that a series *converges*? Remember that we started this section with two principles. First, series are the discrete equivalent of integrals. Second, whenever possible, we want to convert discrete problems into continuous problems.

Example 4.48. Let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The limit of the terms is $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the divergence test doesn't tell us anything. We can use a computer to calculate some experimental values: we compute that $\sum_{i=1}^{10} \frac{1}{i^2} \approx 1.55$ and $\sum_{i=1}^{1000} \frac{1}{i^2} \approx 1.64$. This makes it look like the series is converging; but can we prove it?

Let's draw a picture (figure 4.4). Let $f(x) = \frac{1}{x^2}$, and then the values of the sequence we're adding up are the points $f(n)$. Treat each of these points as the right endpoint of a rectangle of width one; then we see the integral of f from 1 to k is definitely larger than $\sum_{n=2}^{k} \frac{1}{n^2}$. (We did leave out the first term of the series, but that doesn't matter; since it's finite, it can't affect whether our series converges.)

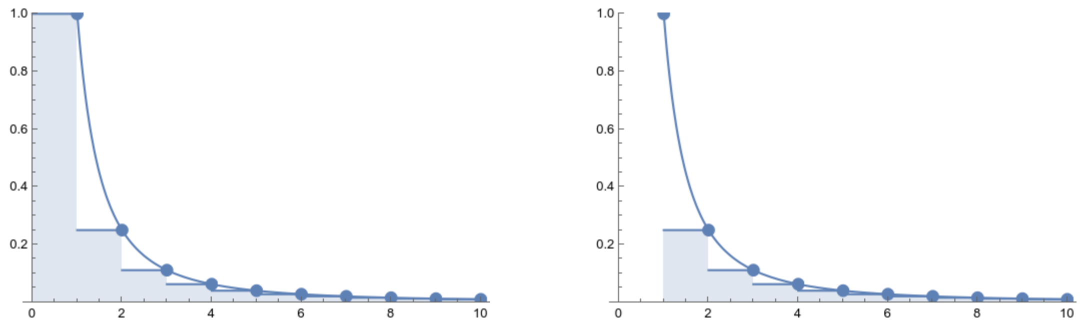

Figure 4.4: Removing the first rectangle changes the value of the integral, but can't affect whether it's finite or infinite.

Thus

$$\sum_{n=2}^{k} \frac{1}{n^2} \leq \int_1^k \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^k = 1 - \frac{1}{k}.$$

Taking the limit gives a right hand side of 1, and thus the sum $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is bounded and so must converge.

Remark 4.49. It turns out that the exact sum of this series is $\pi^2/6 \approx 1.64493$. This was first proven by Leonhard Euler in 1734, originally establishing his reputation. The proof is moderately complicated and requires a number of tools relating to power series, which we will discuss later in the course. (If you're interested, look up the "Basel Problem" on Wikipedia).

Example 4.50. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?

We can use the same rough process and roughly the same picture we just did. By taking rectangles with *left* endpoints, we have

$$\sum_{n=1}^{k} \frac{1}{\sqrt{n}} \geq \int_1^k \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}|_1^k = \sqrt{k} - 1.$$

Taking the limit of both sides shows that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \infty - 1$, and thus increases without bound.

We can build these types of argument into a general rule:

Proposition 4.51 (Integral Test)**.** *Suppose f is a continuous, positive, decreasing function on $[m, +\infty)$ for some m. Let $a_n = f(n)$. Then the series $\sum_{n=m}^{\infty} a_n$ converges if and only if $\int_m^{+\infty} f(x)\,dx$ converges. That is:*

- *If $\int_m^{\infty} f(x)\,dx$ converges then $\sum_{n=m}^{\infty} a_n$ converges.*

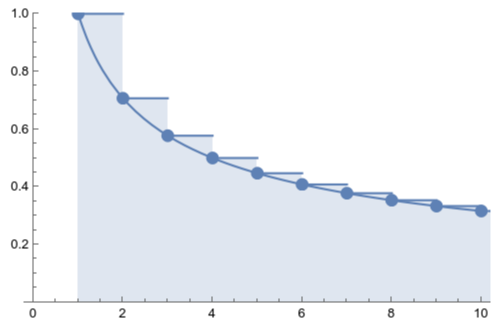

Figure 4.5: The series $\sum \frac{1}{\sqrt{n}}$ diverges, because $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

- *If* $\int_m^\infty f(x)\,dx$ *diverges then* $\sum_{n=m}^\infty a_n$ *diverges.*

Remark 4.52. Note that this doesn't tell us what the sum of the series is, just that it exists. In general, if we want to know the exact sum of a series we need a way to write a closed-form formula for the sequence of partial sums, which is hard. This is what I meant when I said that we don't have a useful equivalent to the fundamental theorem of calculus.

Most of the rest of the tools we'll develop in this class will only be used to establish that some series converges at all. This on its own can be useful, and we'll make it very useful in section 5 when we discuss Power Series and Taylor Series.

Example 4.53. Does $\sum_{n=1}^\infty \frac{2n}{n^2+1}$ converge?

Let $f(x) = \frac{2x}{x^2+1}$. Then f is clearly positive and continuous, and $f'(x) = \frac{2(x^2+1)-4x^2}{(x^2+1)^2}$ is negative so f is decreasing. So we can use the integral test.

$$\int_1^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_1^t \frac{2x}{x^2+1}\,dx$$
$$= \lim_{t\to+\infty} \ln|x^2+1||_1^t = \lim_{t\to+\infty} \ln|t^2+1| - \ln|2| = +\infty.$$

So $\int_1^{+\infty} f(x)\,dx$ diverges, and thus so does $\sum_{n=1}^\infty \frac{2n}{n^2+1}$.

Proposition 4.54. *The series* $\sum_{n=1}^\infty \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Proof. If $p = 1$ this is the harmonic series, and we know it diverges.

If $p \neq 1$ then $f(x) = \frac{1}{x^p}$ is a positive, decreasing, continuous function, so we can use the integral test. We have

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty}\int_1^t x^{-p}\,dx = \lim_{t\to+\infty}\frac{x^{1-p}}{1-p}|_1^t = \lim_{t\to+\infty}\frac{t^{1-p}}{1-p} - \frac{1}{1-p}.$$

This converges precisely when $1-p<0$, precisely when $p>1$. □

Example 4.55. Does $\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+3n^3+n}$ converge?

We could technically use the integral test here. But that would, unfortunately, require us to integrate $\frac{x^2-x}{x^4+3x^3+x}$. This is definitely *possible* using a partial fractions argument, but it's not fun and it's not clean.

But we can try to argue something like this: We know that $n^2 - n < n^2$, and we know that $n^4 + 3n^3 + n > n^4$. This means that

$$\frac{n^2-n}{n^4+3n^3+n} < \frac{n^2}{n^4} = \frac{1}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+3n^3+n} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and that series converges by the p-series test. This implies that our original series also converges!

We did need the integral test to solve this last problem, because we used the integral test to prove the p-series test and used the p-series test there. But this argument allowed us to avoid having to integrate a difficult function.

4.4 The Comparison Tests

The integral test is powerful, and you can in theory answer nearly any question about positive series with the divergence test and the integral test combined. But in practice, the integral test can be really annoying to use, since we have to actually compute integrals. We want to use the work we've already done to avoid having to do more work.

We can do that by comparing new series to old series we've already worked out, systematizing the argument we made in example 4.55.

Proposition 4.56 (Comparison Test)**.** *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. Then:*

- *If* $\sum_{n=1}^{\infty} a_n$ *converges and* $a_n \geq b_n$ *for all (sufficiently large)* n*, then* $\sum_{n=1}^{\infty} b_n$ *converges.*
- *If* $\sum_{n=1}^{\infty} a_n$ *diverges and* $a_n \leq b_n$ *for all (sufficiently large)* n*, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

(This Comparison Test is the discrete analogue of the Comparison Test for improper integrals we saw in section 3.1.3.)

Remark 4.57. Note that this only applies to series with positive terms. If a series has all positive terms, then either it goes to infinity or it converges (as a consequence of completeness, see section 4.1.1). Comparison rules out going to infinity, so the series has to converge.

But if we allow negative terms, there's a third option: oscillating between multiple values. For instance, $\sum \frac{1}{2^n}$ converges, and $-1 \leq \frac{1}{2^n}$ for all n, but $\sum_{n=1}^{\infty}(-1)$ does not converge.

We can rule out oscillation with something like the squeeze theorem, but that requires a lot more work. This comparison test isn't powerful enough to deal with non-positive series.

Using the comparison test requires us to have something to compare our series with. We usually use a power series $\sum n^p$ or a geometric series $\sum ar^{n-1}$.

Example 4.58. Does $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2+n+1}$ converge?

We know that $n^3 \leq n^3 + n^2 + n + 1$, so $\frac{1}{n^3+n^2+n+1} \leq \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ converges, we know that $\sum_{n=1}^{\infty} \frac{1}{n^3+n^3+n+1}$ converges by the Comparison Test.

Example 4.59. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

We could use the integral test, but we can also comment that $\ln n \geq 1$ for $n \geq 3$, so $\frac{\ln n}{n} \geq \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the comparison test.

Example 4.60. Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge or diverge?

The obvious comparison to make is to observe that $\frac{1}{n!} \leq \frac{1}{n}$. But this doesn't help us, because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, and being less than something that goes to infinity doesn't tell us anything. But obviously $\frac{1}{n!}$ is *much* smaller than $\frac{1}{n}$, so we can probably come up with a better comparison.

For $n > 3$, we can work out that that $n! > n^2$: $n! = n(n-1)(n-2)\dots(3)(2)(1) \geq n(n-1)(n-2)$ and $(n-1)(n-2) > n$. Therefore $\frac{1}{n!} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. Thus the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test.

Alternatively: $n! > 2^{n-1}$, and thus $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. But $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series and converges since $r = 1/2 < 1$, so by the comparison test $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

Example 4.61. Does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converge?

This is a lot harder to work with. The obvious comparison is ti $\frac{1}{n^3}$, but it's not actually true that $\frac{1}{n^3-n^2+1} \leq \frac{1}{n^3}$. (In fact, $n^3 > n^3 - n^2 + 1$ for $n > 1$).

We can save it by fiddling with our comparison, and making the series we're comparing to bigger. Instead of $1/n^3$ we can try something like $2/n^3$. And it turns out that $n^3/2 < n^3 - n^2 + 1$ for $n > 1$, since $n^2 < n^3/2 + 1$. So $\frac{1}{n^3-n^2+1} \leq \frac{2}{n^3}$. This shows that $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converges by the comparison test.

This argument worked, but it's fiddly and annoying and seems like it must be too complicated; we'd like to be able to say that $\frac{1}{n^3-n^2+1}$ looks "basically like" $\frac{1}{n^3}$ and so they behave the same. Fortunately there's a way to make that work out.

Proposition 4.62 (Limit Comparison Test)**.** *Suppose* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms, and* $\lim_{n\to+\infty} \frac{a_n}{b_n}$ *exists and is a finite, nonzero number. Then either both series converge, or both series diverge.*

Thus we have
$$\lim_{n\to+\infty} \frac{1/n^3}{1/(n^3-n^2+1)} = \lim_{n\to+\infty} \frac{n^3-n^2+1}{n^3} = 1,$$
and since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$.

Example 4.63. Does $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ converge?

We suspect we can compare this to $\frac{n^2}{n^3}$, or in fact to $\frac{1}{n}$, which has matching top degree. We check by calculating
$$\lim_{n\to+\infty} \frac{\frac{n^2+1}{n^3+n^2+1}}{1/n} = \lim_{n\to+\infty} \frac{n^3+n}{n^3+n^2+1} = \lim_{n\to+\infty} \frac{1+n^{-2}}{1+n^{-1}+n^{-3}} = 1.$$
This is a real number between 0 and $+\infty$. Thus, since $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, by the limit comparison test $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ also diverges.

Example 4.64. Does $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converge or diverge?

The numerator has the order of n and the denominator has the order of $n^{5/2}$, so we want to compare this to $\frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$. So we calculate
$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{n+5}{\sqrt{n^5+n^3+n}}}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{5/2}+5n^{3/2}}{\sqrt{n^5+n^3+n}} \\ &= \lim_{n\to\infty} \frac{1+5/n}{\sqrt{1+1/n^2+1/n^4}} = 1.\end{aligned}$$
This is a real number in $(0,\infty)$, and thus the two series have the same convergence behavior.

Since $3/2 > 1$, by the p-series test we know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converges.

Example 4.65. Does the series $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converge or diverge?

We can't really use the regular comparison test here; the obvious point of comparison is $\sum \frac{1}{3^n}$, but $\frac{1}{3^n-2} > \frac{1}{3^n}$. But we can compute
$$\lim_{n\to\infty} \frac{1/(3^n-2)}{1/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-2} = \lim_{n\to\infty} \frac{1}{1-2/3^n} = 1.$$
Thus by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converges.

We *could* talk a lot more about limit comparison to a geometric series, but there'll be a better way to handle this in section 4.6 when we talk about the ratio test.

4.5 Non-Positive Series

So far we've only discussed series with all positive terms, and we have a pretty good handle on them: we use the integral test to work out some basic examples, and then solve others with the comparison tests.

Things get a little trickier when we want to talk about series that include negative terms. They can get very complicated, but we'll start off with an easy type of example.

4.5.1 Alternating Series

Definition 4.66. An *alternating series* is a series whose terms are alternately positive and negative: either all the odd terms are negative and the even terms are positive, or all the even terms are negative and all the odd terms are positive.

Example 4.67. Some alternating series are

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n+3} = -\frac{1}{4} + \frac{4}{5} - \frac{9}{6} + \frac{16}{7} - \dots$$

Every alternating series $\sum a_n$ looks like $\sum (-1)^n |a_n|$ or $\sum (-1)^{n-1} |a_n|$.

Alternating series are relatively easy to study, because they have such a regular pattern. Fundamentally, an alternating series will go up, and then down, and then up again, but not as high as at first. Each peak will be lower than the previous peak, and each low point will be higher than the previous low point, as wee see in figure 4.6, so the series much converge somewhere.

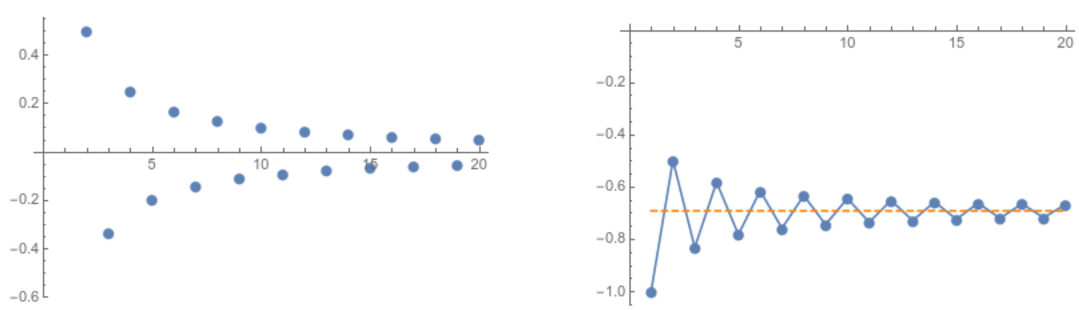

Figure 4.6: When we add up the terms of an alternating series, they oscillate up and down around the limit of the series

Proposition 4.68 (Alternating Series Test)**.** *If* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is an alternating series such that* $b_{n+1} < b_n$ *for all (sufficiently large)* n*, and* $\lim_{n\to+\infty} b_n = 0$*, then the series is convergent.*

Sketch of Proof. The limit $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots \leq b_1$ is bounded above, and $(b_1 - b_2) + (b_3 - b_4) + \ldots$ is increasing, so the sequence of even partial sums $(s_2, s_4, s_6, \ldots)$ must converge to some limit. But s_{2n+1} has to be close to s_{2n}, so the entire sequence must converge. □

Example 4.69. The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the alternating series test, since $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Example 4.70. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}$ does not converge. The series is alternating, but the alternating series test does not apply because $\lim_{n\to+\infty} \frac{n}{n+1} = 1 \neq 0$. In fact, we see that $\lim_{n\to+\infty}(-1)^{n-1}\frac{n}{n+1}$ does not exist, so by the divergence test this series diverges.

Example 4.71. The series $\sum_{n=1}^{\infty}(-1)^{n}\frac{n^3}{n^4+2}$ converges. The sequence $\frac{n^3}{n^4+2}$ is decreasing, as we can see by taking the derivative of $f(x) = \frac{x^3}{x^4+2}$. Further, the limit is zero, so by the alternating series test the series converges.

The Alternating Series Test, combined with the Divergence Test means that we can test the convergence of (almost) any alternating series really easily. If the terms go to zero, it converges by the alternating series test; if the terms don't go to zero, it diverges by the divergence test.

Thus normally the divergence test is a necessary but not sufficient condition. For an alternating series specifically, it is both necessary and sufficient.

One other nice thing about alternating series is that we have a very good estimate of how close we are to the true sum. That means we can calulate estimates fairly easily, and know exactly how many terms we need to work out to be correct within our desired margin of error.

Proposition 4.72 (Alternating Series Estimation)**.** *If* $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is an alternating series that satisfies the hypotheses of the Alternating Series Test, then* $|s - s_n| \leq b_{n+1}$*.*

Sketch of proof. As we saw in figure 4.6, each term we add moves us *past* the limit. So our error at s_n has to be less than the size of the move we'll make by adding on the next term b_{n+1}. □

Example 4.73. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. What is the error term in approximating the sum if we calculate the first ten terms?

The size of the error is smaller than the next term, which is the eleventh term, which is $\frac{1}{11}$. Thus $\sum_{n=1}^{10} \frac{(-1)^n}{n} \approx -.65$ is within $\frac{1}{11}$ of the infinite sum. In section 5.2.1 we will see that the exact sum of this series is $-\log 2 \approx -.69$.

Example 4.74. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. What is the error term in approximating the sum if we calculate the first ten terms?

The size of the error is smaller than the next term, which is the eleventh term, which is $\frac{1}{121}$. Thus $\sum_{n=1}^{10} \frac{(-1)^n}{n^2} \approx -.818$ is within $\frac{1}{121}$ of the infinite sum, which turns out to be about $-.822\ldots$.

Example 4.75. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2n+1}$. How many terms do we have to calculate to get the answer to within 1/100?

The ninth term has size $\frac{1}{9^2+18+1} = \frac{1}{100}$, so we need to compute the first eight terms. This gives approximately $-.1720$, while the true answer is approximately $-.1775$.

4.5.2 Absolute Convergence

The alternating series test allowed us to study one particular type of series with non-positive terms, but there are many non-positive series that aren't alternating. It's very difficult to study them in general, but the idea of absolute convergence allows us to mostly duck the question.

Definition 4.76. A series $\sum a_n$ is called *absolutely convergent* if $\sum |a_n|$ converges.

A series $\sum a_n$ is *conditionally convergent* if it is convergent but not absolutely convergent.

The series $\sum |a_n|$ is always non-negative, so we can use our tools from sections 4.3 and 4.4 to figure out whether this absolute-value series converges. But is that useful?

Theorem 4.77. *If $\sum a_n$ is absolutely convergent, then it converges.*

Proof. $0 \leq a_n + |a_n| \leq 2|a_n|$, and $a_n + |a_n| \geq 0$. We have $\sum_{n=1}^{\infty} 2|a_n|$ converges, so by comparision test $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} a_n$$

is a difference of convergent series and so converges. □

Remark 4.78. The converse is not true! $\sum_{n=1}^{\infty} (-1)^n/n$ is convergent (by the alternating series test) but not absolutely convergent. This is why it's possible, and in fact relatively common, for a series to be conditionally convergent.

This theorem lets us study many sequences with positive and negative terms.

Example 4.79. The series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent. We have $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test $\sum \left|\frac{\sin n}{n^2}\right|$ converges.

Example 4.80. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So the series is conditionally convergent.

Example 4.81. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the divergence test, since $\lim_{n \to +\infty} \sin n$ does not exist.

These are the three possible answers we can ever have: absolute convergence, conditional convergence, and divergence.

Example 4.82. We claim that $\sum_{n=1}^{\infty} (-1)^n/n^2$ converges absolutely. For $\sum_{n=1}^{\infty} |(-1)^n/n^2| = \sum_{n=1}^{\infty} n^{-2}$ which we know converges.

The main purpose of this is to take questions about series with some negative terms, and turn them into questions about series with positive real terms, so that our previous tests apply.

Example 4.83. Does the series $\sum_{n=1}^{\infty} \frac{\sin(n^2+e^n)}{n^2}$ converge?

We have that $\left|\frac{\sin(n^2+e^n)}{n^2-n}\right| = \leq \frac{1}{n^2}$, so by the comparison test this series converges absolutely. Thus it converges.

As one final note: absolutely convergent series are much nicer and easier to handle than series that are merely conditionally convergent.

Proposition 4.84. *If a series is absolutely convergent, then the sum doesn't depend on the order of the terms. (In particular, the sum of a series of positive numbers doesn't depend on the order of the terms).*

If a series is conditionally convergent but not absolutely convergent, then the sum does *depend on the order of the terms; and in fact by reordering the terms we can get essentially any sum we like.*

More precisely, the Riemann Series Theorem says that if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series, then by reordering we can cause the sum to converge to any real number, or to diverge to $+\infty$ or $-\infty$.

Example 4.85. It's possible to compute that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$. But we also have

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{\ln 2}{2}.$$

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers. Rewrite it as $\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} c_n$ where the b_n are all the positive terms and the c_n are all the negative terms. If both of these sums converged, then the series would converge absolutely (since $\sum b_n + \sum c_n = \sum b_n + c_n = \sum |a_n|$); if one converged and the other diverged, then $\sum a_n$ would diverge. So $\sum b_n = \sum c_n = +\infty$.

Pick a target M. Arrange the sum as follows: include positive terms until the sum is above M. Then include negative terms until the sum is below M. Repeat, alternating, infinitely. The sum will oscillate around M and converge to M.

If we want the sum to approach $+\infty$, include positive terms until the sum is above 1, then a negative term, then positive terms until the sum is above 2, then a negative term, and so on. □

4.6 The Ratio and Root Tests

Once we know to look for absolute convergence, we can use the comparison test on any series, but we'd like to cut out some steps.

4.6.1 The Ratio Test

If we imagine comparing our series to a geometric series, we get the *ratio test*:

Proposition 4.86 (Ratio Test). *If $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then:*

- *If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
- *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark 4.87. If $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ or does not exist, then the ratio test tells us nothing. We have to use some other test or technique.

This test tends to work well when our series looks "almost" geometric, meaning the terms have nth powers in them, or when the terms contain factorials. It works badly when the terms have additions and subtractions within them, or more generally when the terms look polynomial rather than exponential.

Example 4.88. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Since it has a factorial, this is a natural place to apply the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$

so by the ratio test this series converges absolutely.

Example 4.89. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Again, there are factorials so we want to use the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}\right| = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n.$$

It's maybe not immediately clear to us whether this converges, or to what. But we know that

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

by our definition of e from section 1.2.1. Thus we have

$$\lim \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n} = \frac{1}{e} < 1.$$

So by the ratio test this series converges absolutely.

Example 4.90. What about $\sum_{n=1}^{\infty} \frac{r^n}{n!}$? For what r does it converge?

We still want to use the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{r^{n+1}/(n+1)!}{r^n/n!}\right| = \lim_{n\to\infty} \frac{r}{n+1} = 0 < 1.$$

By the ratio test, this converges absolutely for any r.

Example 4.91. Now let $r > 0$ be a real number. Does $\sum_{n=1}^{\infty} \frac{n!}{r^n}$ converge or diverge?

This is similar but opposite to the previous problem. We have

$$\lim_{n\to\infty} \left|\frac{(n+1)!/r^{n+1}}{n!/r^n}\right| = \lim_{n\to\infty} \frac{n+1}{r} = +\infty > 1$$

so by the ratio test this diverges.

Example 4.92. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$.

We compute

$$\lim_{n\to\infty} \left|\frac{((n+1)^2+1)/(2^{n+1})}{(n^2+1)(2^n)}\right| = \lim_{n\to\infty} \frac{n^2+2n+1}{(n^2+1)\cdot 2} = \frac{1}{2} < 1.$$

So by the ratio test this converges.

4.6.2 The Root Test

The Root Test is similar to the ratio test, but is sometimes slightly easier or harder to apply than the Ratio Test is.

Proposition 4.93 (Root Test). *If* $\sum_{n=1}^{\infty} a_n$ *is a series and* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$*, then:*

- *If* $L < 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
- *If* $L > 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark 4.94. If $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1$ or does not exist, then the root test tells us nothing. We have to use some other test or technique.

This is most useful when our series has an nth power of some polynomial involving n. The root test works well if each term is a perfect nth power, and poorly if we have something like $n2^n$ where some terms aren't covered by the exponent. The ratio test works well if our terms don't have any additions or subtractions in them, but do have exponents.

Example 4.95. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+1}{2n+2}\right)^n$.

We have $a_n = \left(\frac{5n+1}{2n+2}\right)^n$ and thus $\sqrt[n]{|a_n|} = \frac{5n+2}{2n+2}$. So $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \frac{5}{2} > 1$ so the series converges absolutely.

Example 4.96. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{3n^2+2n+1}\right)^n$

Our terms are perfect nth powers, so the root test seems natural. We compute

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2n^2+1}{3n^2+2n+1}\right|^n} = \lim_{n\to\infty} \frac{2n^2+1}{3n^2+2n+1} = \frac{2}{3} < 1.$$

So by the Root Test this series converges absolutely.

Example 4.97. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$.

Our terms are perfect nth powers, so we can try the root test. We compute

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{n+1}\right|^n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

so the root test doesn't tell us anything! We could try the ratio test, but it would be much harder to apply and would give the same answer—the root and ratio tests always give the same answer.

We could try either a comparison test or an integral test, but the integral seems nasty, and I'm not sure what to compare it to. And at this point we realize we forgot the first rule of series convergence: try the divergence test first! We have

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} \neq 0.$$

So by the divergence test, this series diverges.