6 Parametrization and Coordinates

In this section we're going to dabble a bit in working with multiple variables. We'll specifically going to look at the way we can describe some plane geometry ideas that we haven't thought much about in the past. You can view this as a teaser for multivariable calculus; but it also gives you a couple of basic ideas that are useful to have in your pocket if you don't actually take multi.

6.1 Curves and Motion

In this section we want to study curves in the plane. By a curve we mean, essentially, any shape that is in some sense "one-dimensional". So a line, a circle, and a curving spiral space are all curves.

The essence of a curve is the one-dimensionality. We capture this idea by requiring position on our curves to be described by one single real number. That is, we can describe our position on the curve with exactly one coordinate. We say a system of coordinates for an object is a "parametrization", because it describes the object with some number of parameters.

Definition 6.1. We say a pair of equations $x = f(t), y = g(t)$ is a *parametrization of a curve* in the plane.

Sometimes you'll see these functions just called $x(t)$ and $y(t)$. It's common to refer to the pair with its own name: we can write $\vec{r}(t) = (f(t), g(t))$.

Example 6.2. Let's find a parametrization for the curve $y = x^2$.

We see that we can parametrize this by the function $x = t, y = t^2$. You'll notice that this is basically the original function formula: we have $x = t$ and $y = t^2 = x^2$. Any time we have a curve that is the graph of a function, we effectively have a parametrization for free; the input variable gives us a parametrization.

Example 6.3. Let's parametrize a circle of radius 1. Notice that we can't use the same trick as last time, since this isn't a function.

We could try something like $f(t) = t, g(t) = \sqrt{1 - t^2}$ for $-1 \leq t \leq 1$. This sort of works, but only captures the top half of the circle. We could keep trying to make this idea work, but it basically won't.

Instead, we take advantage of the fact that circles are fundamentally trigonometric. We see that $x = \cos(t), y = \sin(t)$ will give us every point on the circle—in fact, this is the usual unit circle definition of sin and cos. In particular, we have $(f(0), g(0)) = (1, 0)$ is the rightmost point of the circle, and as t increases we move counterclockwise around the circle.

However, this isn't the only possible parametrization. For instance, we could instead take $x = \sin(t), y = \cos(t)$. This will still parametrize the circle, but it starts at $(x, y) = (0, 1)$ which is the top of the circle, and proceeds clockwise. So we get the same *shape* but a different *path*.

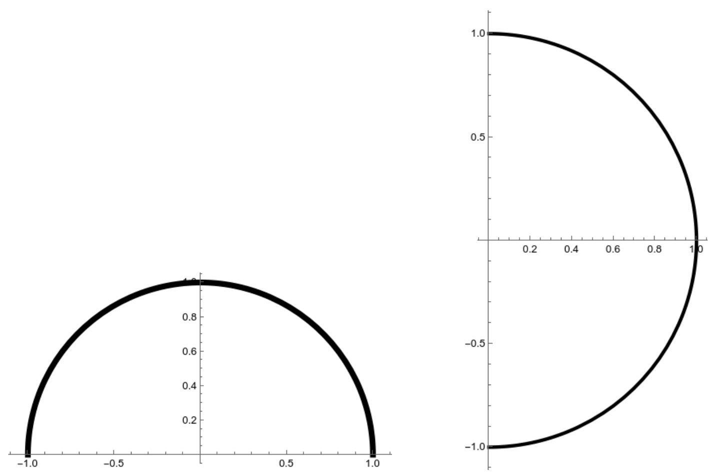


Figure 6.1: The graphs of $(\cos(t), \sin(t))$ and $(\sin(t), \cos(t))$ for $0 \leq t \leq \pi$.

Example 6.4. Another nice property of parametrizations is that it's easy to shift them in space. Let's parametrize a circle of radius 2 centered at $(3, 2)$, going counterclockwise starting from the right-hand point.

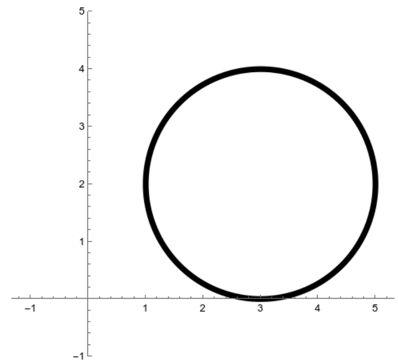
We know that a circle of radius 1 centered at the origin is $\vec{r}(t) = (\cos(t), \sin(t))$. To get radius 3, we multiply by 3; then to shift the center, we add $(3, 2)$, leaving us with the parametrization $\vec{r}(t) = (3 + 2\cos(t), 2 + 2\sin(t))$.

If we want to start from left-hand point and go clockwise, we can do a couple things. One is to flip the circle upside down and start halfway around; this would give $\vec{r}(t) = (3 + 2\cos(t + \pi), 2 - 2\sin(t + \pi))$.

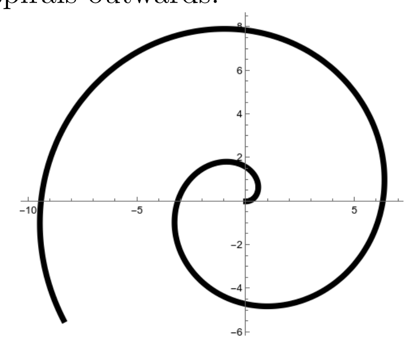
Alternatively, we could start from the parametrization $(\sin(t), \cos(t))$, which already goes clockwise. Then we would get $\vec{r}(t) = (3 + 2\sin(t - \pi/2), 2 + 2\cos(t - \pi/2))$.

In general, choices of parametrization aren't unique. Often we can make a problem easier (or harder) by changing our choice of coordinates.

Example 6.5. Let's consider the curve given by $x = 5\cos t, y = 5\sin t$. This gives us a circle of radius 5.



We can make this more interesting by taking something like $x = t\cos(t), y = t\sin(t)$. This will create a shape that spirals outwards.



We can also make more fun shapes with parametrization.

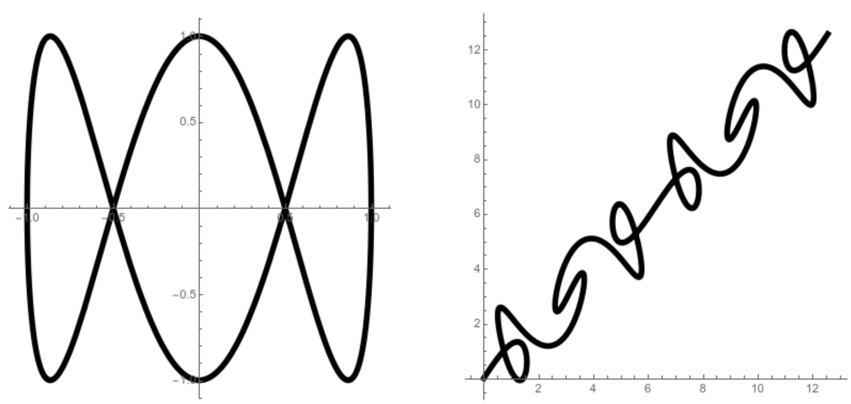


Figure 6.2: $\vec{r}(t) = (\cos(t), \sin(3t))$ and $\vec{s}(t) = (t + \sin(3t), t + \sin(5t))$

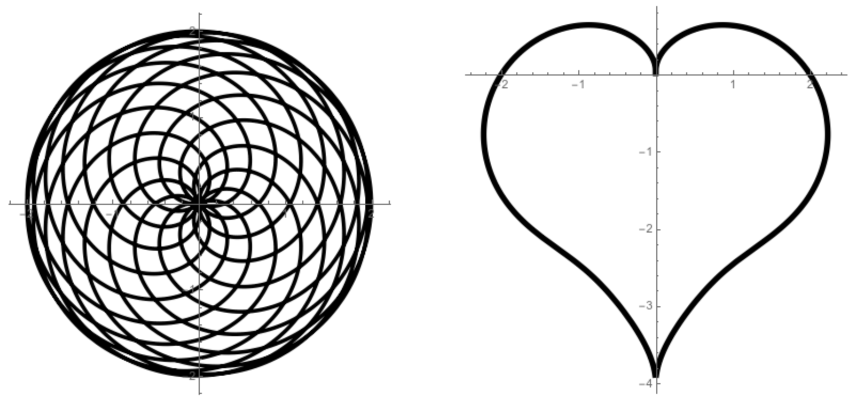


Figure 6.3: $\vec{r}(t) = (\cos(10t) + \cos(21t), \sin(10t) + \sin(21t))$ and
$\vec{s}(t) = \left(\cos(t)\left(\frac{\sin(t)\sqrt{|\cos(t)|}}{\sin(t)+7/5} - 2\sin(t) + 2\right), \sin(t)\left(\frac{\sin(t)\sqrt{|\cos(t)|}}{\sin(t)+7/5} - 2\sin(t) + 2\right)\right)$

I'm always surprised that complicated-looking shapes sometimes have very simple parametrizations, whereas simple shapes like the heart have sometimes very complicated curves. This is closely related to how much "like a circle" the shape is. Some shapes fall naturally out of throwing together elementary functions, and others do not.

It turns out that for any curve, it's possible to find a parametrization using the magic of *Fourier series*. But the formulas tend to wind up looking pretty ridiculous.

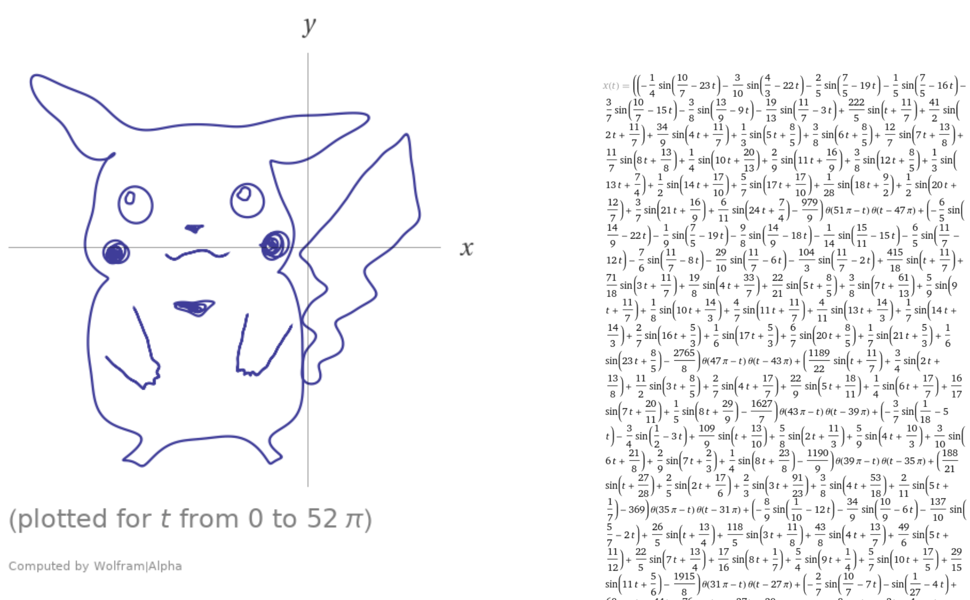


Figure 6.4: See more examples at `https://www.wolframalpha.com/examples/mathematics/geometry/curves-and-surfaces/popular-curves/`

6.1.1 Calculus of Curves

So far we've discussed parametric equations as giving position as a function of time, and talking about the direction and sometimes the speed of motion. As in the single-variable case, we can make this more precise by the theory of derivatives.

Speed is change in position with respect to time. We can define this pretty easily:

Definition 6.6. The *velocity* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

What is this really doing? We're just defining $\vec{r}\,'(t)$ to be the amount that $\vec{r}(t)$ changes by if we increase t by a little bit, which is just how the normal derivative works. Notice that $\vec{r}\,'(t)$ has two pieces, the x component and the y component. These represent the amount of change in the x coordinate and the y coordinate, respectively. Together we call them the *tangent vector* to $\vec{r}(t)$ at a point. We can define a *parametric tangent line* by looking at the function

$$\vec{T}(t) = \vec{r}(a) + \vec{r}\,'(a)(t-a).$$

The

Just like the definition of a normal derivative, this definition is a bit hard to work with. In the single variable case we came up with a bunch of rules we could use to compute derivatives. Here we don't need to go through all of that again: the x and y coordinates change independently, so we can consider them independently. (This is implicitly because derivatives are always linear, so we can write the derivative of a sum as the sum of the derivatives).

We really just have two single-variable derivatives. And we already know how to compute those.

Proposition 6.7. *Let $\vec{r}(t) = (f(t), g(t))$ be a parametrization. Then*

$$\vec{r}\,'(t) = (f'(t), g'(t)).$$

Proof.

$$\begin{aligned}\vec{r}\,'(t) &= \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} \\ &= \lim_{h\to 0} \frac{(f(t+h),g(t+h))-(f(t),g(t))}{h} \\ &= \lim_{h\to 0} \left(\frac{f(t+h)-f(t)}{h}, \frac{g(t+h)-g(t)}{h}\right) \\ &= (f'(t), g'(t)).\end{aligned}$$

□

Example 6.8. Consider the circle parametrized by $(\cos(t), \sin(t))$. Then the derivative is $\vec{r}\,'(t) = (-\sin(t), \cos(t))$.

If we want to find the tangent vector at the point $(1,0)$, we compute the derivative and plug in $t = 0$, so we get $\vec{r}\,'(0) = (0,1)$ as your vector, and the tangent line is

$$(1,0) + (0,1)(t-0) = (1, 0+t).$$

Now suppose want the tangent line at $(\sqrt{2}/2, \sqrt{2}/2)$. This occurs at time $t = \pi/4$ and so we compute $\vec{r}\,'(\pi/4) = (-\sqrt{2}/2, \sqrt{2}/2)$. Thus the tangent line is

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)(t-\pi/4) = \sqrt{2}/2(1-t, 1+t).$$

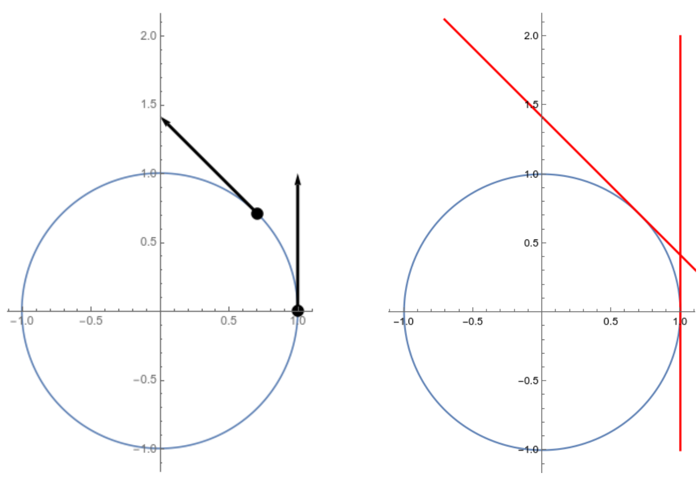


Figure 6.5: Left: the tangent vectors. Right: the tangent lines.

After taking the first derivative, we can also take the second (and further) derivatives. As in the single variable case, if the function gives position, and the derivative gives velocity, then the second derivative gives acceleration.

Definition 6.9. The *acceleration* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2} = \lim_{h\to 0} \frac{\vec{r}\,'(t+h) - \vec{r}\,'(t)}{h}.$$

As you'd expect, we can compute the acceleration just by taking the componentwise second derivatives: we have

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = (f''(t), g''(t)).$$

Example 6.10. Consider again the circle parametrized by $\vec{r}(t) = (\cos(t), \sin(t))$. Then we know that $\vec{r}\,'(t) = (-\sin(t), \cos(t))$, and thus the second derivative is $\vec{r}\,''(t) = (-\cos(t), -\sin(t))$.

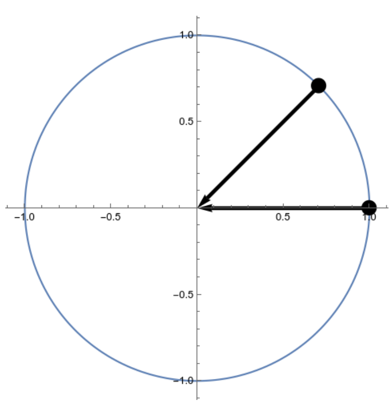
Then we compute that $\vec{r}\,''(0) = (-1, 0)$ and $\vec{r}\,''(\pi/4) = (-\sqrt{2}/2, -\sqrt{2}/2)$.

We notice (figure 6.6 that the acceleration arrows in a circle always point inwards! This is because the motion is at a constant speed, and so the acceleration is only changing direction; so we can't speed up or slow down in the direction of our velocity, and our acceleration must be perpendicular to our velocity.

If we want, we can use the second (and further) derivatives to build an analogue of the Taylor series. For instance, using the second derivative we can get parabolic approximations to our circle. We wind up with the parabolas of figure 6.6

$$(1,0) + (0,1)(t-0) + (-1,0)\frac{1}{2}(t-0)^2 = \left((1 - \frac{t^2}{2}, t\right)$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)(t - \pi/4) + \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)\frac{1}{2!}(t-\pi/4)^2$$
$$= \sqrt{2}/2\left(1 - t - \frac{t^2}{2}, 1 + t - \frac{t^2}{2}\right).$$



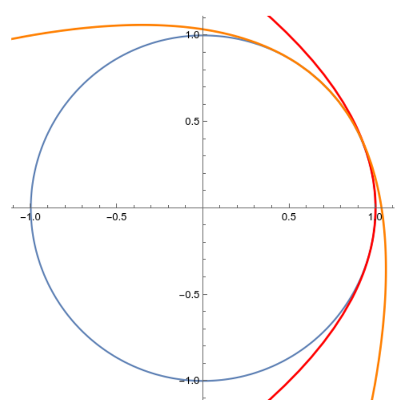


Figure 6.6: Left: the acceleration vectors. Right: the second-order approximations

However, these parametrizations have subtleties that aren't just their shapes.

Example 6.11. Suppose we have the function $\vec{r}(t) = (2 + 4t^3 + 4t, 6 + 3t^3 + 3t)$. Then we can compute the velocity to be $\vec{r}\,'(t) = (12t^2 + 4, 9t^2 + 3)$, and the acceleration is given by $\vec{r}\,''(t) = (24t, 18t)$.

But if we graph the function, it just looks like a line! We have $r(t) = (2, 6) + (t^3 + t)(4, 3)$, so we're always on the line with slope $3/4$. Then why is the velocity so variable? We're constantly maintaining the same *direction*, but the *speed* in which we move in that direction changes, as we see in figure 6.7.

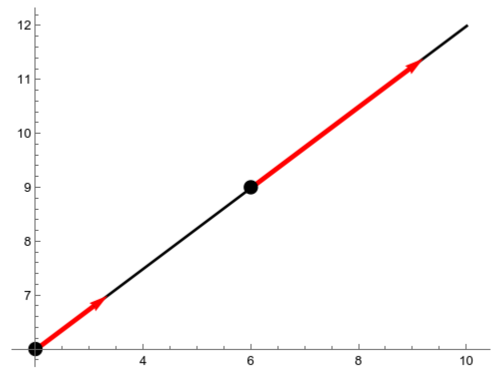


Figure 6.7: (Scaled) velocity vectors starting at $(2, 6)$ and at $(6, 9)$

We can also compute the arc length for parametrized curves just like we can for regular curves. Remember in section 3.2.1 we saw that the arc length of a curve was

$$\begin{aligned} L &= \int_{a_1}^{b_1} ds \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \end{aligned}$$

(where the bounds of the integral depend on the variable we're integrating with respect to, so x goes from a to b as y goes from c to d).

Now suppose we have a parametrized curve $(x, y) = (f(t), g(t))$. We can take α, β so that $f(\alpha) = a$ and $f(\beta) = b$, so that as t goes from α to β we have x going from a to b. Then a change of variables gives

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt}\, dt$$
$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This works even if our curve isn't given by writing y as a function of x; we have the same basic argument that an infinitesimal line segment has length roughly

$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t.$$

Then we can add up the lengths of each short line segment, and take the limit as the number of segments goes to zero.

Example 6.12. We can parametrize a circle by $(x, y) = (\cos(t), \sin(t))$ as t varies from 0 to 2π. Then we have

$$L = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$
$$= \int_0^{2\pi} 1\, dt = 2\pi.$$

What if we instead let t vary from 0 to 4π? We get

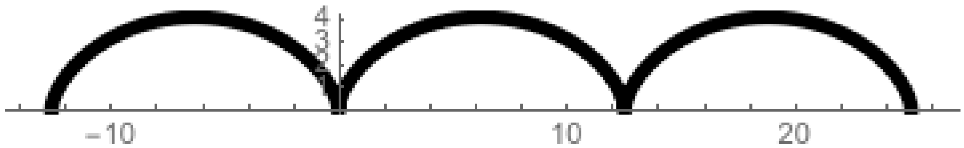
$$L = \int_0^{4\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$
$$= \int_0^{4\pi} 1\, dt = 4\pi.$$

But the apparent curve hasn't changed! We get twice as much arc length because we're traveling around the circle twice; this calculation finds the distance actually covered by the parametrization, even if you're repeating the same path multiple times.

Finally, we can find the area under a parametric curve, in a kind of silly way. We know that area is given by the integral $A = \int_a^b y\, dx$. If we have a parametrization $y = g(t)$ and $x = f(t)$, then we can do a u-substitution $x = f(t)$. Then $dx = f'(t)\, dt$ and we get

$$A = \int_a^b y\, dx = \int_a^b g(t) f'(t)\, dt.$$

Example 6.13. Consider the curve $(x, y) = \big(2(t - \sin(t)), 2(1 - \cos(t))\big)$.



If we want the area under one arch, we have

$$\int_0^{4\pi} y\,dx = \int_0^{2\pi} 2(1-\cos(t)) \cdot \frac{d}{dt} 2(t - \sin(t))\,dt$$
$$= \int_0^{2\pi} 4(1 - \cos^2(t))\,dt = 12\pi.$$

6.2 Polar Coordinates

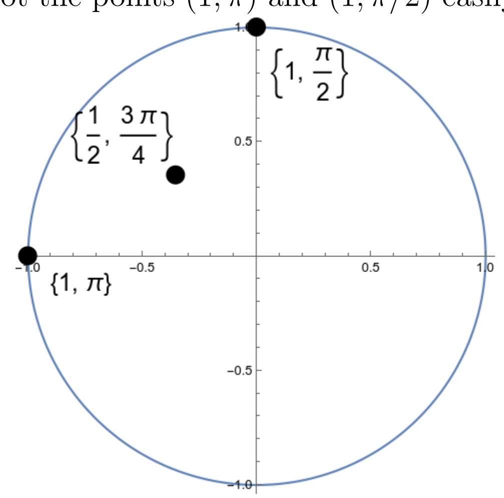
We usually work in a coordinate system known as *rectilinear* or *Cartesian* coordinates (after the French mathematician and philosopher René Descartes, who invented the idea of describing shapes with equations). These coordinates are often very useful, but sometimes they're not the best choice; for instance, they handle circles a bit awkwardly. It's much easier to describe a circle or circle-like region in a new coordinate system based on circular motion.

Definition 6.14. The *polar coordinates* of a point P in the plane are a pair of numbers (r, θ), where r is the distance between P and the origin O, and θ is the angle between the positive x-axis and a line from the origin to P.

We usually choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

(Your textbook allow negative r coordinates, which is important for dealing with certain types of equations, but usually isn't very helpful.)

Example 6.15. We can plot the points $(1, \pi)$ and $(1, \pi/2)$ easily. What about $(1/2, 3\pi/4)$?



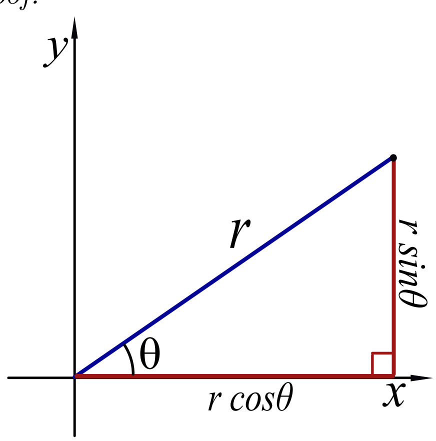
We see that the same point can have many different names in polar coordinates; in fact, $(r, \theta) = (r, \theta + 2\pi)$ for any point (r, θ). We generally either choose to have θ in $[-\pi, \pi]$, or in $[0, 2\pi)$. But if it's convenient you can choose it to have any size.

It's useful to be able to convert between polar coordinates and cartesian coordinates. But some simple trigonometry makes that easy.

Proposition 6.16. *Suppose (x, y) are the cartesian coordinates of a point P, and $(r\, theta)$ are the polar coordinates. Then:*

- $x = r \cos\theta$
- $y = r \sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \pm \arctan y/x$.

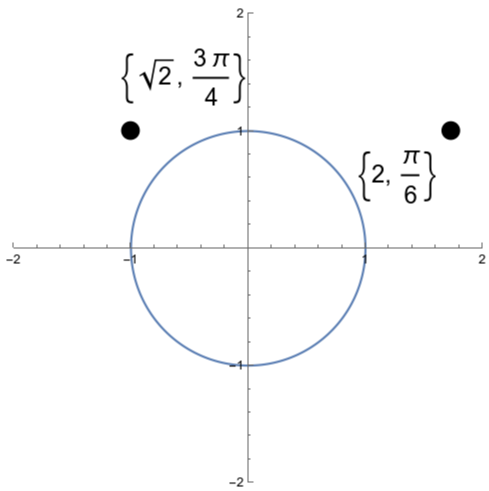
Proof.



□

Example 6.17. If we convert $(2, \pi/6)$ to Cartesian coordinates, we get $(\sqrt{3}, 1)$.

If we convert $(-1, 1)$ to polar coordinates, we get $(\sqrt{2}, 3\pi/4)$. Alternatively, we could say $(\sqrt{2}, -5\pi/4)$ or $(\sqrt{2}, 11\pi/4)$. Polar coordinates have multiple labels for the same point.



We can also graph curves in polar coordinates.

Example 6.18. Circular shapes tend to be easier to describe in polar coordinates.

- The polar equation for a circle of radius c is as simple as possible: it's just $r = c$. The closed disk of radius c is given by the set $\{(r, c) : 0 \leq r \leq c, 0 \leq \theta < 2\pi\}$. The Cartesian coordinates are $\{(x, y) : x^2 + y^2 \leq c^2\}$.
- What does the equation $\theta = 1$ look like? It's a line starting at the origin and going up and to the right at a $\pi/4$ or $45°$ angle. (If we allow r to be negative, the line extends in both directions.)

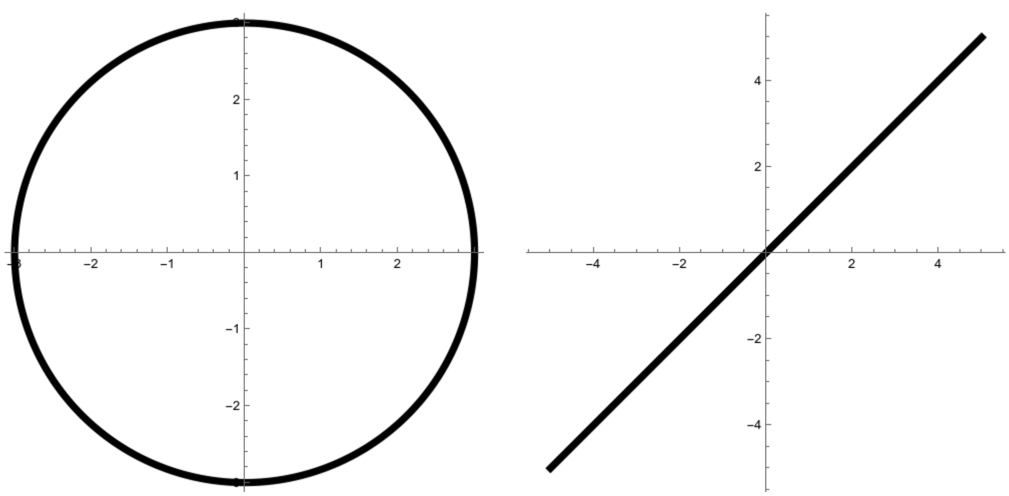


Figure 6.8: Left: The graph of $r = 3$. Right: the graph of $\theta = 1$.

- The wedge of the closed unit disk in the first (upper-right) quadrant is $\{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$. The Cartesian coordinates are $\{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1.\}$

- The set $\{(r,\theta) : 1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2\}$ is a wedge of an annulus with inner radius 1 and outer radius 2, in the third (lower-left) quadrant. The Cartesian coordinates here are $\{(x,y) : x \leq 0, y \leq 0, 1 \leq x^2 + y^2 \leq 4\}$.

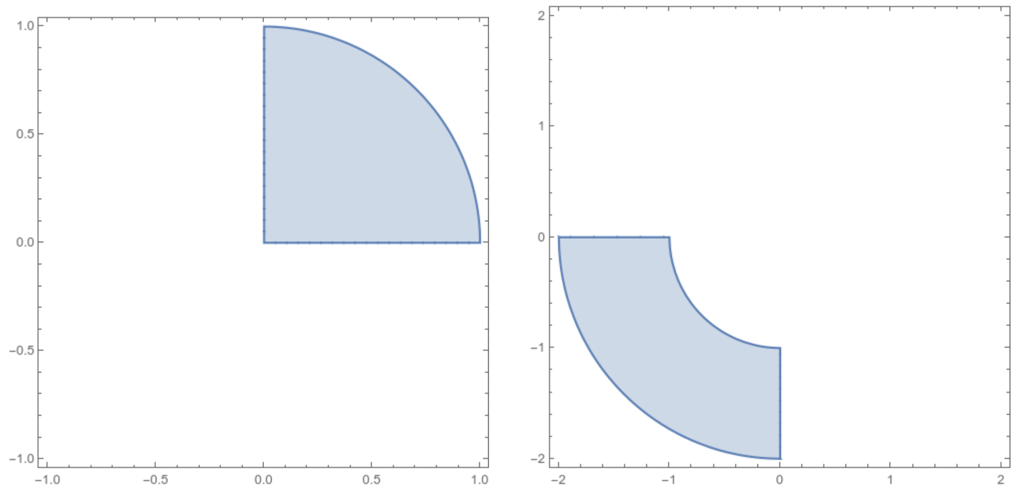


Figure 6.9: Left: The region $0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$. Right: the region $1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2$.

Lines, on the other hand, tend to be obnoxious in polar. Through the origin they're not too bad:

Example 6.19. The polar equation for the line $y = 2x$ is $r \sin\theta = 2r\cos\theta$, which reduces to $\sin\theta = 2\cos\theta$. With a little more work, you can compute that this is equvalent to the line $\theta = \arctan(2)$.

But lines that don't go through the origin tend to produce genuinely obnoxious equations.

Example 6.20. Consider the line $y = 2x + 1$. Substituting in according to the rules in proposition 6.16 gives us $r\sin\theta = 2r\cos\theta + 1$. solving for r gives $r = \frac{1}{\sin(\theta) - 2\cos(\theta)}$.

And here are some much less obvious examples.

Example 6.21. $r = 2\cos\theta$ gives a circle centered at $(1,0)$. In general, $r = 2n\cos(\theta)$ gives a circle centered at (n,θ). We can actually work this out from the substitution rules in proposition 6.16:

$$
\begin{aligned}
r &= 2\cos(\theta) \\
\sqrt{x^2+y^2} &= 2\cos(\arctan(y/x)) \\
= 2\frac{x}{\sqrt{x^2+y^2}} \\
x^2+y^2 &= 2x \\
(x-1)^2+y^2 &= 1.
\end{aligned}
$$

Example 6.22. $r = 1 + \cos(\theta)$ gives a shape called a *cardioid*. $r = \cos(2\theta)$ gives a flower. You can kind of work out why this would happen, but mostly they're just kinda pretty.

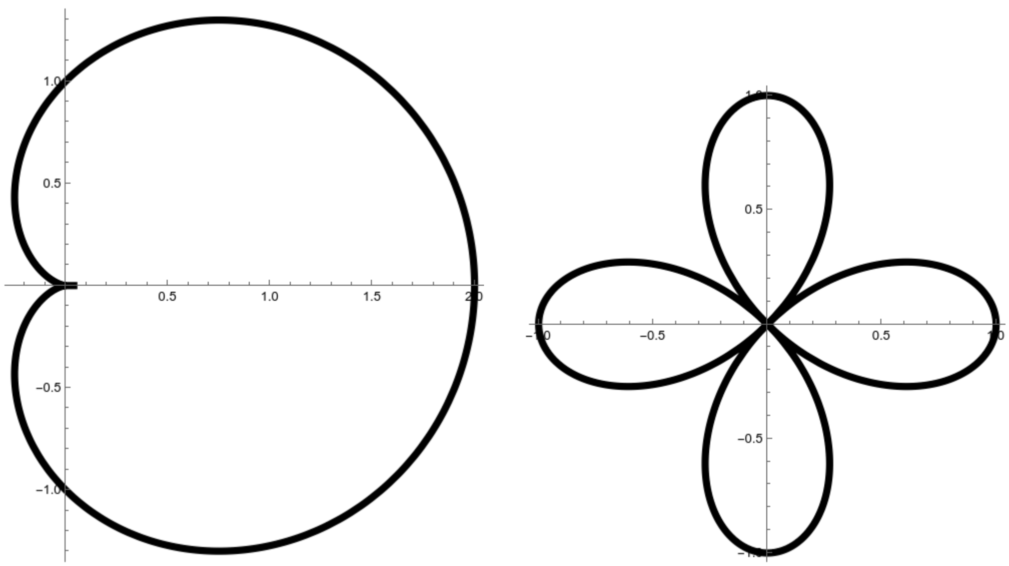


Figure 6.10: Left: $r = 1 + \cos\theta$. Right: $r = \cos(2\theta)$.

6.2.1 Derivatives in Polar Coordinates

We can take the derivative of a polar function, but there's nothing really new there. The derivative measures the rate at which the radius r changes as the angle θ changes.

Example 6.23. Suppose we have $r = 2\cos\theta$. Then $\frac{dr}{d\theta} = -2\sin\theta$.

More interesting is looking for the equations of tangent lines. To do this we want to fall back on our theory of parametrization from section 6.1.1 to find a slope. We have formulas for x and y as (multivariable) functions of r and θ; but since here r is a function of θ, we can write everything in terms of θ.

Example 6.24. Suppose we want to find an equation for the tangent line to $r = 2\cos\theta$ at the point $r = \sqrt{3}, \theta = \pi/6$, which translates to $x = 3/2, y = \sqrt{3}/2$. We know that

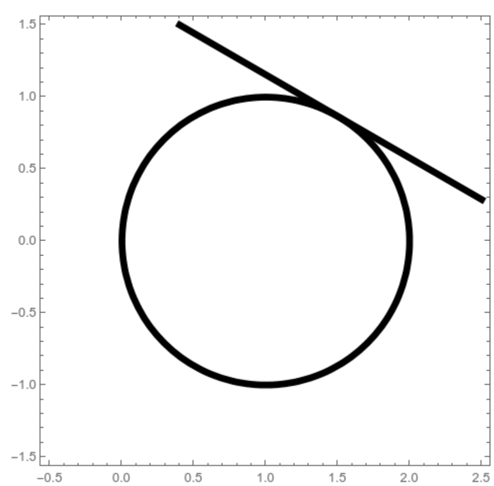
$$\begin{aligned} x &= r\cos\theta = 2\cos^2\theta \\ y &= r\sin\theta = 2\cos\theta\sin\theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin^2(\theta) + 2\cos^2(\theta)}{-4\cos\theta\sin\theta} \\ &= \frac{\sin^2(\theta) - \cos^2(\theta)}{2\cos(\theta)\sin(\theta)} \\ \frac{dy}{dx}(\pi/6) &= \frac{1/4 - 3/4}{2 \cdot \sqrt{3}/2 \cdot 1/2} = \frac{-1}{\sqrt{3}} \end{aligned}$$

so the slope of our tangent line is $\frac{1}{\sqrt{3}}$ and the (Cartesian) equation is

$$y - \sqrt{3}/2 = \frac{-1}{\sqrt{3}}(x - 3/2).$$

If we really want to we can turn this into a polar equation:

$$r\sin\theta - \sqrt{3}/2 = \frac{-1}{\sqrt{3}}(r\cos(\theta) - 3/2).$$



6.2.2 Polar Integrals: areas and lengths

We've seen that polar coordinates tend to make circular equations become much simpler than their cartesian equivalents, but lines (and anything else rigid and rectangular) become much more complex.

We want to exploit this complexity reduction to make integrals of functions over circular regions easier. When we integrated over a rectangular region, we did this by dividing the

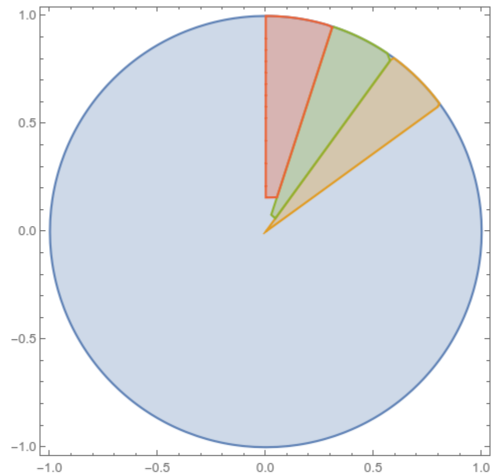


Figure 6.11: The subwedges for $2\pi/10 \leq \theta \leq 3\pi/10$, $3\pi/10 \leq \theta \leq 4\pi/10$, and $4\pi/10 \leq \theta \leq 5\pi/10$

region into rectangles. Using polar coordinates to integrate over a circular or wedge-like region, we'll divide the region into *subwedges*, as seen in in figure 6.11

What is the area of a wedge? If the wedge has outer radius r and is spanned by an angle $\Delta\theta$, then it is $\frac{\Delta\theta}{2\pi}$ of a circle of radius r. The area of such a circle is πr^2, so the area of our wedge is

$$\frac{\Delta\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \Delta\theta.$$

Thus if we have $r = f(\theta)$ on a shape going from $\theta = a$ to $\theta = b$, we can subdivide our region into n rectangles, and then our area is approximately

$$A \approx \sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta.$$

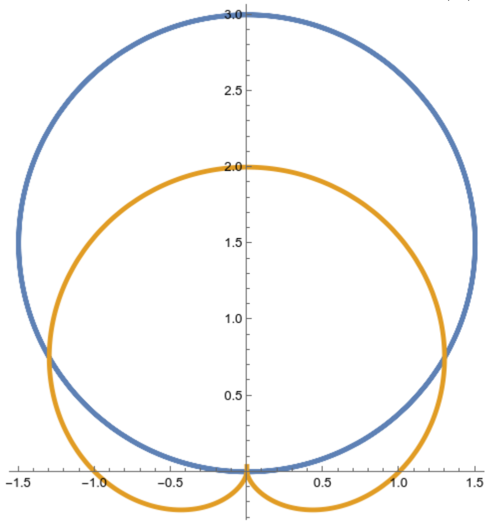
You should recognize this as a Riemann sum! Now that we have the explicit sum we no longer need to worry too much about where it came from; we know that Riemann sums become integrals as n approaches ∞, so we get

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta \qquad \text{or} \qquad A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$

Example 6.25. Let's find the area inside one petal of the flower $r = \cos(2\theta)$. This is the range as θ goes from $-\pi/4$ to $\pi/4$. So we have the integral

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2(2\theta)\,d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(1+\cos(4\theta)\,d\theta \\ &= \frac{\theta}{4} + \frac{\sin(4\theta)}{16}\Big|_{-\pi/4}^{\pi/4} \\ &= \frac{\pi}{16} + 0 - \frac{-\pi}{16} - 0 = \frac{\pi}{8}. \end{aligned}$$

Example 6.26. Find the area of the region inside $r = 3\sin(\theta)$ and outside $r = 1 + \sin\theta$.



These two curves intersect when

$$\begin{aligned} 3\sin(\theta) &= 1 + \sin\theta \\ 2\sin\theta &= 1 \\ \sin\theta &= 1/2 \\ \theta &= \pi/6 \text{ or } 5\pi/6. \end{aligned}$$

We can find the area inside the blue curve $r = 3\sin\theta$ and then subtract off the area inside the yellow curve, so we get

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta \\ &= \left(\frac{9\sqrt{3}}{8} + \frac{12\pi}{8}\right) - \left(\frac{9\sqrt{3}}{8} + \frac{4\pi}{8}\right) \\ &= \frac{12\pi}{8} - \frac{4\pi}{8} = \pi. \end{aligned}$$

Notice something subtle is going on here: the first inegral doesn't compute the entire area of the blue circle, which would just be π. That's because we're only counting the area between $\theta = \pi/6$ and $\theta = 5\pi/6$; we're ignoring the thing slices at the bottom of the circle on either side.

Arc Length We can also compute the lengths of arcs in polar coordinates.

$$\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)\right)^2}\, d\theta \\
&= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.
\end{aligned}$$

Example 6.27. Find length of cardioid $1 + \sin\theta$.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\
&= \int_0^{2} \pi\sqrt{2 + 2\sin\theta}\, d\theta \\
&= 8.
\end{aligned}$$