Math 1232: Single-Variable Calculus 2
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Recitation 0

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What is a *function*?

Solution: A rule that takes an input and assigns a specific output.

Sometimes we have a function and we want to undo it. That is, we want to know the output and find the input. This is a basically reasonable question: "What do I have to do if I want to get a specific result" is something we find ourselves thinking a lot. But now we want to look at the mathematical side of it.

Definition 0.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

Example 0.2.

- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.

Problem 1.

(a) Can you find an inverse for $f(x) = x^2$?

(b) Can you find an inverse for $f(x) = x^3$?

(c) What makes these two functions different?

(d) What needs to happen for us to be able to invert or "undo" a function?

(e) What would this look like on a graph? [Hint: the answer is not the vertical line test, but thinking about the vertical line test will help you.]

Solution:

(a) Nope!

(b) Sure. $g(x) = \sqrt[3]{x}$.

(c) We can't possibly undo $f(x) = x^2$. Because if we ask, say, "what do we need to square to get 4", there are actually two answers.

(d) We can't undo a function if there are two inputs that give the same output. Because if that happens, we can never figure out the input just by knowing the output.

(e) The vertical line test checks whether something is a function: that is, it checks whether any single input has two different outputs. But here we want the opposite: we need to check if any *output* has two different *inputs*. So instead of the vertical line test, we can use the horizontal line test.

Definition 0.3. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Any invertible function has to be one-to-one. Less obviously, any one-to-one function is invertible.

Problem 2. (a) Is the function $f(x) = |x|$ one-to-one? Prove it is, or find a counterexample.

(b) Is the function $g(x) = 5x^3 + 3$ one-to-one? Prove it is, or find a counterexample.

(c) Find an inverses for any of these functions that were one-to-one.

Solution:

(a) No, because $f(-1) = 1 = f(1)$.

(b) Yes. Suppose $g(x) = g(y)$. Then we have

$$\begin{aligned} 5x^3 + 3 &= 5y^3 + 3 \\ 5x^3 &= 5y^3 \\ x^3 &= y^3 \\ \sqrt[3]{x^3} &= \sqrt[3]{y^3} \\ x &= y. \end{aligned}$$

(If you want to be clever, you can get to the line $x^3 = y^3$ and then remember the function x^3 is one-to-one

(c) Now we want to solve the equation $y = g(x)$.

$$\begin{aligned} y &= 5x^3 + 3 \\ y - 3 &= 5x^3 \\ \frac{y-3}{5} &= x^3 \\ \sqrt[3]{\frac{y-3}{5}} &= x \end{aligned}$$

So the inverse is

$$g^{-1}(y) = \sqrt[3]{\frac{y-3}{5}}.$$

Problem 3. Consider the function $f(x) = x^4$.

(a) Is this one-to-one? Is it invertible?

(b) Then what is $\sqrt[4]{x}$?

(c) What needs to happen for $\sqrt[4]{x}$ to be an inverse?

(d) Can you find a completely different set of numbers where f is invertible?? Find an inverse on that domain.

Solution:

(a) $f(-2) = 16 = f(2)$, so this function is not one-to-one. That means it's not invertible.

(b) This is *trying* to be an inverse but it's not. Why not? Because $\sqrt[4]{2^4} = 2$, but $\sqrt[4]{(-2)^4} = 2 \neq -2$.

(c) This only works if you start with a positive number. So f is one-to-one if you restrict it to the domain $[0, +\infty)$. (There are lots of choices but this is the most obvious.)

(d) On this domain, the inverse is $\sqrt[4]{x}$.

(e) If you want to be trolly, you could say something like $[1, +\infty)$, or $[0, 5)$, or any number of other choices.

But to be *completely* different we want to flip things around. we'll say the domain is $(-\infty, 0]$. On this domain, the inverse is $-\sqrt[4]{x}$.

This is why there are two fourth roots of any positive number. (Over the complex numbers there are in fact four, but we don't have do deal with that yet.)

Problem 4. Let $f(x) = x^5 + x$.

(a) Is this function one-to-one? You won't be able to prove it directly from the definition, but you can use calculus to make a clear argument.

(b) Can you find an inverse for this function?

(c) Can you find $f^{-1}(2)$? $f^{-1}(34)$? $f^{-1}(-2)$?

(d) Can you find $(f^{-1})'(2)$?

(e) Can you find $(f^{-1})'(34)$? $(f^{-1})'(-2)$?

Solution:

(a) Yes! We see that $f'(x) = 5x^4 + 1 \geq 1$, so the function is always increasing. That means it can't repeat, and so must be one-to-one.

(b) No! I mean this in a fairly robust way. If we go ask Wolfram Alpha to find the inverse to this function, it gives the answer

$$x\,{}_2F_2\left(\frac{3 \pm 1}{10}, \frac{7 \pm 1}{10}; \frac{5}{4}, \frac{5 \pm 1}{8}; -\frac{3125x^4}{256}\right).$$

I don't know what that means, either, but it's not helpful. (It is apparently a "hypergeometric pfq".)

If we ask Mathematica to solve the equation $y = x^5 + x$ we get the even more wonderful answer that x is the solution to the polynomial $x^5 + x - y = 0$. None of this is helpful.

There definitely is an inverse. But you can't find it and neither can I. (This is a theorem.)

(c) We don't have a formula, but we can still find these answers by guess-and-check. Plugging in small numbers gives

$$f(0) = 0 \qquad f(1) = 2 \qquad f(2) = 34$$
$$f(-1) = -2 \qquad f(-2) = -34.$$

Thus $f^{-1}(2) = 1$, and $f^{-1}(34) = 2$, and $f^{-1}(-2) = -1$.

(d) The inverse function theorem tells us that

$$\begin{aligned}(f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} \\ &= \frac{1}{f'(1)} = \frac{1}{5(1)^4+1} = \frac{1}{6}.\end{aligned}$$

(e) Again,

$$\begin{aligned}(f^{-1})'(34) &= \frac{1}{f'(f^{-1}(34))} \\ &= \frac{1}{f'(2)} = \frac{1}{5(2)^4+1} = \frac{1}{81} \\ (f^{-1})'(-2) &= \frac{1}{f'(f^{-1}(-2))} \\ &= \frac{1}{f'(-1)} = \frac{1}{5(-1)^4+1} = \frac{1}{6}.\end{aligned}$$