Math 1232: Single-Variable Calculus 2

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Recitation 1

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Problem 1. Consider $f(x) = \cos(x)$.

(a) Is this function one-to-one? Why or why not?

(b) What domains can you restrict it to to get a one-to-one function?

(c) What value "should" you pick to solve $\cos(x) = 0$? What about $\cos(x) = 1$? $\cos(x) = -1$?

(d) What domain should you pick to create an inverse?

Solution:

(a) No. $\cos(0) = 1 = \cos(2\pi)$.

(b) Looking at a graph, we need to go from a peak to a trough or a trough to a peak. So we need something that looks like $[n\pi, (n+1)\pi]$ for some integer n.

(c) This is pretty subjective. We definitely want $\cos(0) = 1$. It makes most sense to me to take $\cos(\pi/2) = 0$ and $\cos(\pi) = -1$, but you could maybe argue for $\cos(-\pi/2)$ and $\cos(-\pi)$ instead.

(d) Consequently we want to define cosine on $[0, \pi]$ to get a one-to-one function.

Problem 2. In class we talked about the function $f(x) = x^5 + x$. This function is 1-1, but we can't write down an inverse for it. But there are still some things we can say about the inverse.

(a) Can you find $f^{-1}(2)$? $f^{-1}(34)$? $f^{-1}(-2)$?

(b) Can you find $(f^{-1})'(2)$?

(c) Can you find $(f^{-1})'(34)$? $(f^{-1})'(-2)$?

Solution:

(a) We don't have a formula, but we can still find these answers by guess-and-check. Plugging in small numbers gives

$$\begin{aligned} f(0) &= 0 & f(1) &= 2 & f(2) &= 34 \\ & & f(-1) &= -2 & f(-2) &= -34. \end{aligned}$$

Thus $f^{-1}(2) = 1$, and $f^{-1}(34) = 2$, and $f^{-1}(-2) = -1$.

(b) The inverse function theorem tells us that

$$\begin{aligned} (f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} \\ &= \frac{1}{f'(1)} = \frac{1}{5(1)^4+1} = \frac{1}{6}. \end{aligned}$$

(c) Again,

$$\begin{aligned} (f^{-1})'(34) &= \frac{1}{f'(f^{-1}(34))} \\ &= \frac{1}{f'(2)} = \frac{1}{5(2)^4+1} = \frac{1}{81} \\ (f^{-1})'(-2) &= \frac{1}{f'(f^{-1}(-2))} \\ &= \frac{1}{f'(-1)} = \frac{1}{5(-1)^4+1} = \frac{1}{6}. \end{aligned}$$

Problem 3. Let $g(x) = \sqrt[3]{x^3 + x + 6}$.

(a) Can you compute an inverse for g?

(b) Can you find $\left(g^{-1}\right)'(2)$?

(a) The function is invertible, since it's increasing. You even, in theory, could find the inverse. But realistically you're not going to; the formula is:

$$g^{-1}(y) = \frac{\sqrt[3]{\frac{2}{3}}}{\sqrt[3]{-9y^3 + \sqrt{3}\sqrt{27y^6 - 324y^3 + 976} + 54}} - \frac{\sqrt[3]{-9y^3 + \sqrt{3}\sqrt{27y^6 - 324y^3 + 976} + 54}}{\sqrt[3]{2} \cdot 3^{2/3}},$$

and I wouldn't expect anyone to successfully find or work with that.

(b) This is much easier. By the Inverse Function Theorem we know that

$$\begin{aligned}(g^{-1})'(x) &= \frac{1}{g'(g^{-1}(x))} \\ g'(x) &= \frac{1}{3}(x^3 + x + 6)^{-2/3}(3x^2 + 1).\end{aligned}$$

We just need to find $g^{-1}(2)$, which we can, essentially, solve by guessing and checking: and it turns out that $g(1) = 2$, so $g^{-1}(2) = 1$. So we have

$$\begin{aligned}g'(1) &= \frac{1}{3}(1^3 + 1 + 6)^{-2/3}(3(1)^2 + 1) = \frac{1}{3}8^{-2/3}(4) = \frac{1}{3} \\ (g^{-1})'(x) &= \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(1)} = 3.\end{aligned}$$

Problem 4. (a) Consider the functions $f(x) = x^3 - x^2 + x$ and $g(x) = x^3 - x^2 - x$. Which one is invertible and why?

(b) Consider the functions $f(x) = 3^x + x$ and $g(x) = 3^x - x$. Can you figure out which one is invertible?

Solution:

(a) We might try guess-and-check; in that case we might see that $g(1) = -1 = g(-1)$, and thus g isn't one-to-one or invertible.

For a more systematic approach, we can compute derivatives. We see that

$$\begin{aligned}f'(x) &= 3x^2 - 2x + 1 \\ g'(x) &= 3x^2 - 2x - 1\end{aligned}$$

$g'(0) = -1 < 0$ but $g'(2) = 7 > 0$ (and in fact $g'(1) = 0$). This tells us that g goes down and then back up, and so it will fail the horizontal line test.

In contrast, with a little work we can see that $f'(x) \geq 0$ for all values of x. For instance we know that $x^2 - 2x + 1 = (x-1)^2 \geq 0$ and thus $f'(x) = 2x^2 + (x-1)^2 \geq 0$. So f is always increasing, and that means that it is one-to-one.

(b) This would be easy if we knew how to compute the derivative of 3^x, but we don't yet. (Soon!)

But we *do* know that 3^x is an increasing function, and so $3^x + x$ is adding two increasing functions together and so still increasing. So this must be one-to-one.

If we look at $g(x)$, we can try plugging some numbers in. $g(0) = 1, g(1) = 2, g(2) = 7$, which all seems increasing. But $g(-1) = 1/3 + 1 = 4/3$, and so g decreases then increases again; and in particular we know that for some value of x between 0 and 1, $g(x) = 4/3$. So g is not one-to-one.

Problem 5. (a) Compute $\log_3(6) + \log_3(9/2)$.

(b) Compute $\log_4(8) - \log_4(2)$.

(c) Rewrite the expression $\log_5(15) + \log_5(75) - \log_5(12)$ as an integer plus a logarithm.

(d) Solve $e^{5-3s} = 10$.

Solution:

(a) $\log_3(6) + \log_3(9/2) = \log_3(6 \cdot 9/2) = \log_3(27) = 3$.

(b) $\log_4(8) - \log_4(2) = \log_4(4) = 1$.

Alternatively $\log_4(8) - \log_4(2) = 1.5 - .5 = 14$.

(c)

$$\begin{aligned}
\log_5(15) + \log_5(75) - \log_5(12) &= \log_5(15 \cdot 75/12) \\
&= \log_5\left(\frac{3}{4} \cdot 125\right) \\
&= \log_5(125) + \log_5(3/4) = 3 + \log_5(3/4).
\end{aligned}$$

You could also write this as $3 + \log_5(3) - \log_5(4)$ if you want.

(d) The problem here is that there's a variable in the exponent. To deal with difficult exponents, we take a logarithm. Then we see that $5 - 3x = \ln 10$ and so $x = \frac{5 - \ln 10}{3}$.