Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 10
Due Monday, November 4

This week's mastery quiz has two topics. Everyone should submit M3. If you already have a 2/2 on S7, you don't need to submit it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$. (Does it converge or diverge?)

Solution: We compute

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx \\ &= \lim_{t\to\infty} \frac{1}{2}\arctan(x^2)\Big|_1^t = \lim_{t\to\infty} \frac{1}{2}\left(\arctan(t^2) - \arctan(1)\right) \\ &= \frac{1}{2}\left(\pi/2 - \pi/4\right) = \frac{\pi}{8}.\end{aligned}$$

This is finite and convergent, so by the integral test, the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

Alternatively, we could observe that $\frac{n}{n^4+1} \leq \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-series test, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges by the comparison test.

(b) Analyze the convergence of the series $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$. (Does it converge or diverge?)

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{x\to+\infty} \frac{x^{\nearrow\infty}}{\ln(x)_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to+\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

(c) Analyze the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. (Does it converge or diverge?)

Solution: We could try the divergence test, but $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$ so it passes. This is smaller than the harmonic series: we could do a comparison test and note that $\frac{1}{n\ln(n)} \leq \frac{1}{n}$. But this doesn't help us, because the harmonic series diverges. In fact, no reasonable comparison will help us here.

But we can compute

$$\begin{aligned}\int_2^{+\infty} \frac{1}{x\ln(x)}\,dx &= \lim_{t\to+\infty} \int_2^t \frac{1}{x\ln(x)}\,dx \\ &= \lim_{t\to+\infty} \ln(\ln(x)) \mid \Big|_2^t \\ &= \lim_{t\to+\infty} \ln(\ln(t)) - \ln(\ln(2)) = +\infty.\end{aligned}$$

This integral diverges, so the series diverges by the integral test.

Name: Recitation Section:

S7: Sequences and Series

(a) Consider the sequence $(a_n) = (3, {}^6/_2, {}^9/_6, {}^{12}/_{24}, {}^{15}/_{120}, \dots)$. Find a formula for the nth term a_n. Compute $\lim_{n\to\infty} a_n$.

Solution: We have $a_n = \frac{3n}{n!}$, and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{n!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} =$

Solution: This is a geometric series with $a = 9/5$ and $r = 9/5$. We have $r > 1$, so we know that it diverges.

(c) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

Solution: We can do a partial fractions decomposition: we have

$$\begin{aligned} 2 &= A(n+1) + B(n+3) & \\ 2 &= 2B & \Rightarrow B = 1 \\ 2 &= -2A & \Rightarrow A = -1 \end{aligned}$$

so our sum is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

More rigorously, we have

$$\begin{aligned} \sum_{n=1}^{k} \frac{2}{n^2+4n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\ &\qquad + \dots + \left(\frac{1}{k+1} - \frac{1}{k+3}\right) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\ \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$