Math 1232: Single-Variable Calculus 2
George Washington University Fall 2024
Recitation 10

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Problem 1. Analyze the convergence of the following series. Write clean arguments that establish whether they diverge, converge conditionally, or converge absolutely. Think about what tools/tests you want to use, and why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{\ln(n)}$

Solution:

(a) Here we can start with the divergence test. We know that

$$n! = n(n-1)(n-2)\dots(3)(2)(1) \geq n(2)(2)\dots(2)(2)1 = n2^{n-2}$$

and thus $\frac{n!}{2^n} \geq \frac{n}{4}$. Thus

$$\lim_{n\to\infty} \frac{n!}{2^n} \geq \lim_{n\to\infty} \frac{n}{4} = +\infty \neq 0$$

and so this series diverges by the divergence test.

(b) First we consider absolute convergence. We know that $\frac{1}{\ln(n)} \geq \frac{1}{n}$ when $n \geq 3$, and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges. So by the comparison test $\sum_{n=3}^{\infty} \frac{1}{\ln(n)}$ diverges, and so our original series does not converge absolutely.

Now we consider conditional convergence. $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$, and thus by the Alternating Series Test, our series converges.

Therefore our original series converges conditionally.

Problem 2 (Bessel Function)**.** The Bessel function (of order 0) is critical to any physics done in cylindrical coordinates, and thus any physics that occurs on a cylinder. We saw it earlier as the solution to the differential equation $x^2y'' + xy' + x^2y = 0$, but it can also be given by the power series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

What is the radius of convergence? What is the interval of convergence?

Solution: We use the ratio test. We have $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}
\lim \left| \frac{a_{n+1}}{a_n} \right| &= \lim \left| \frac{x^{2n+2}/2^{2n+2}((n+1)!)^2}{x^{2n}/2^{2n}(n!)^2} \right| \\
&= \lim \left| \frac{x^{2n+2}}{x^{2n}} \frac{2^{2n}}{2^{2n+2}} \frac{(n!)^2}{((n+1)!)^2} \right| \\
&= \lim \frac{|x|^2}{4(n+1)^2} = 0.
\end{aligned}$$

Thus the Bessel function of order 0 converges absolutely for all real numbers x. We say the radius of convergence is ∞ and the interval is all reals, or $(-\infty, +\infty)$.

Problem 3. What is the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x-1)^n}{7^{n+2}}?$$

Solution: Using the ratio test, we have

$$\lim \left| \frac{(n+1)^2(x-1)^{n+1}/7^{n+3}}{n^2(x-1)^n/7^{n+2}} \right| = \lim \frac{|x-1|}{7} \frac{(n+1)^2}{n^2} = \frac{|x-1|}{7}.$$

So the series converges absolutely when $|x-1| < 7$, and thus on the interval $(-6, 8)$. For the full interval we need to test the endpoints, at $x = -6$ and $x = 8$.

When $x = -6$ we have

$$\sum \frac{n^2(-7)^n}{7^{n+2}} = \sum (-1)^n \frac{n^2}{49}.$$

This is an alternating series, but the terms tend towards infinity and so by the divergence test it diverges.

Similarly, when $x = 8$ we have

$$\sum \frac{n^2 7^n}{7^{n+2}} = \sum \frac{n^2}{49}.$$

The terms tend towards infinity, so the series diverges by the divergence test.

Thus the real interval of convergence is $(-6, 8)$.

Problem 4 (Bonus)**.** In class we showed that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges to a number between 1 and 1/2. I claimed that it converged to $\ln 2$. But this convergence is conditional, and that matters.

(a) Write out the first twelve terms of this series.

(b) Reorganize them so that you have the same collection of numbers add one and then subtract two, then add one, then subtract two, and so on. (You'll have some extras left over and that's fine; remember you have an infinite list of terms.)

(c) What does each triplet look like? Can you simplify that somehow so it looks like something we recognize? (Hint: what happens if you combine the first two terms of a triplet?)

(d) Can you figure out what this sequence of partial sums converges to?

Solution:

(a)
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$$

(b) We get
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$
The 7, 9, and 11 terms are left over, but if we kept going we'd get $\frac{1}{7} - \frac{1}{14} - \frac{1}{16}$ and so on.

(c) We see that each triplet has the pattern $\frac{1}{n} - \frac{1}{2n} - \frac{1}{2n+2}$, where n is an odd number. If we collect the first two terms, we get $\frac{1}{2n} - \frac{1}{2n+2}$. So the first few triplets are
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12}$$

(d) This is exactly half of our original series. So when we add them up in this order, we get $\frac{\ln(2)}{2}$.