Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 11
Due Monday, November 11

This week's mastery quiz has two topics. Everyone should submit both.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n+1}$

Solution: We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{{(-1)^{n+1}3^{n+1}}/{5^{n+1}+1}}{{(-1)^n 3^n}/{5^n+1}} \right| &= \lim_{n\to\infty} \frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)} \\ &= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1} \\ &= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.\end{aligned}$$

This limit is less than 1, so by the ratio test this converges absolutely.

(b) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution: This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful: $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty} \frac{{n}/{n^2+1}}{{1}/{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

(c) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+n}$

Solution: We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{{(-2)^{n+1}}/{(n+1)^3+n+1}}{{(-2)^n}/{n^3+n}} \right| &= \lim_{n\to\infty} \frac{2(n^3+n)}{(n+1)^3+n+1} \\ &= \lim_{n\to\infty} 2 > 1.\end{aligned}$$

This limit is greater than 1, so by the ratio test this diverges.

Alternatively, we could note that

$$\lim_{n\to\infty} \frac{(-2)^n}{n^3+n} = \pm\infty,$$

so by the divergence test this diverges. But it's a little tricky to cleanly argue that this goes to infinity; we can't really use L'Hospital's rule without getting the negative sign out of there somehow.

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2}$.

Solution: We use the ratio test.

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(2x-5)^{n+1}/(n+1)^2}{(2x-5)^n/n^2} \right| &= \lim_{n\to\infty} \left| \frac{(2x-5)n^2}{(n+1)^2} \right| \\ &= |2x-5| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |2x-5|.\end{aligned}$$

So we need $|2x-5| < 1$ or $-1 < 2x-5 < 1$, or $4 < 2x < 6$ or $2 < x < 3$. So the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(4-5)^n}{n^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(6-5)^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

converges by p-series test

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}(x+2)^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{((n+1)!)^2(x+2)^{n+1}/(3n+3)!}{(n!)^2(x+2)^n/(3n)!} \right| = \lim_{n\to\infty} |x+2| \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \quad \leq \frac{|x+2|}{n} = 0$$

for any x. So the radius of convergence is infinity, and this converges for all x.