Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 12
Due Monday, November 18

This week's mastery quiz has three topics. Everyone should submit M4. If you have a 4/4 on M3, or a 2/2 on S8, you don't have to submit them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2+1}$

Solution: We can work this out with the integral test. We have

$$\int_1^{\infty} xe^{-x^2+1}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2+1}\,dx \quad = \lim_{t\to\infty}\frac{-1}{2}e^{-x^2+1}\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}e^2 - \frac{1}{2}e^{-t^2+1} = e^2/2 < \infty.$$

Since this integral converges, the series must also converge by the integral test. Since the series is all positive, it converges absolutely.

Alternatively, we could use the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)e^{-n^2-2n}}{ne^{-n^2+1}} &= \lim_{n\to\infty}\frac{n+1}{n}\frac{e^{n^2-1}}{e^{n^2+2n}}\\ &= \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{e^{2n+1}} = 0.\end{aligned}$$

Since $0 < 1$, this converges absolutely by the Ratio Test.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{2n+3}$

Solution: This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty}\frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{n}{2n+3} = 1/2.$$

Since the series $\sum\frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum\frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

(c) Analyze the convergence of the series $\displaystyle\sum_{n=2}^{\infty}\frac{\ln(n)+n}{n^2-1}$

Solution: You can't really use the limit comparison test here, at least not easily, because the numerator is a bit over-complicated. But you can use the usual comparison test. We know that $n \leq n + \ln(n)$ and $n^2 - 1 < n^2$, so

$$\frac{\ln(n) + n}{n^2 - 1} \geq \frac{n}{n^2} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

M4: Taylor Series

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered at 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}
\frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty} (-4x)^n \\
\frac{2x^2}{1+4x} &= 2x^2 \sum_{n=0}^{\infty} (-4)^n x^n \\
&= \sum_{n=0}^{\infty} 2 \cdot (-4)^n x^{n+2} \\
\text{(or)} \quad &= \sum_{n=2}^{\infty} 2^{2n-3} (-1)^n x^n.
\end{aligned}$$

The radius of convergence is 1/4. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty} \left| \frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n} \right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x, \pi)$ for this function. (That is, find the terms up through the degree four term.)

Solution:

$$\begin{aligned}
f(x) &= \cos^2(x) & f(\pi) &= 1 \\
f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0 \\
f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2 \\
f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0 \\
f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8.
\end{aligned}$$

So we have

$$T_4(x,\pi) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4.$$

(c) If $f(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}(x+2)^n$, compute $\frac{d}{dx}f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx}f(x) = \sum_{n=0}^{\infty} \frac{3^n}{(n-1)!}(x+2)^{n-1}\text{(or much much better)} \quad = \sum_{n=1}^{\infty} \frac{3^n}{(n-1)!}(x+2)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}(x+2)^{n+1} + C$$

$$\text{(or)} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{n!}(x+2)^n + C.$$

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}(n(x-3))^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}(x-3)^{n+1}}{n^n(x-3)^n}\right| = \lim_{n\to\infty}|x-3|\left(\frac{n+1}{n}\right)^n(n+1) \quad \geq |x-3|\lim_{n\to\infty}(n+1) = \infty$$

unless $x = 3$. So the radius of convergence is 0, and the series converges if and only if $x = 3$.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}\frac{(5x-3)^n}{\sqrt{n}}$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}}\right| = \lim_{n\to\infty}\left|\frac{(5x-3)\sqrt{n}}{\sqrt{n+1}}\right|$$

$$= |5x-3|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3| < 1$ or $-1 < 5x-3 < 1$, or $2 < 5x < 4$ or $2/5 < x < 4/5$. We need to have x in the interval $(3/5-1/5, 3/5+1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(2-3)^n}{\sqrt{n}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by p-series test

Thus the interval of convergence is $[2/5, 4/5)$.