Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 13
Due Monday, December 2

This week's mastery quiz has three topics. Everyone should submit M4 and S9. If you have a 4/4 on M3, you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series

Name:

Recitation Section:

M3: Series Convergence

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n}{(n^2)/5 + 3n}$

Solution: This clearly converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2/5-3n} = 0$, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n}{n^2/5 + 3n} \right| = \sum_{n=4}^{\infty} \frac{1}{n^2/5 + 3n},$$

so we can use the Limit Comparison Test to $\frac{1}{n^2}$. We compute

$$\lim_{n\to\infty} \frac{\frac{1}{n^2/5+3n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2/5 + 3n} = 1/5.$$

This is a nonzero real number, so since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges, by the Limit Comparison Test, $\sum_{n=4}^{\infty} \frac{1}{n^2/5+3n}$ converges. Thus our original series converges absolutely. (And thus we don't actually need to check for whether the alternating series test applies.)

(b) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n^4 - 1}{n^5 + 1}$.

Solution: We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. But

$$\lim_{n\to\infty} \frac{3n^4 - 1/n^5 + 1}{1/n} = \lim_{n\to\infty} \frac{3n^5 - n}{n^5 + 1} = 3$$

which is a finite non-zero number, so by the limit comparison test, $\sum_{n=1}^{\infty} \frac{3n^4-1}{n^5+1}$ diverges. So the series does not converge absolutely.

However, $\lim_{n\to\infty} \frac{3n^4-1}{n^5+1} = 0$, so by the alternating series test, $\sum_{n=1}^{\infty} (-1)^n \frac{3n^4-1}{n^5+1}$ converges. So the series converges conditionally.

(c) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 5^n}$.

Solution: We compute that

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}/(n+1)^2 5^{n+1}}{(-3)^n/n^2 5^n} \right| \\ &= \lim_{n\to\infty} \frac{3^{n+1} 5^n n^2}{3^n 5^{n+1} (n+1)^2} \\ &= \lim_{n\to\infty} \frac{3n^2}{5(n+1)^2} = \frac{3}{5}. \end{aligned}$$

Since $L < 1$, by the ratio test this series converges absolutely.

Name: Recitation Section:

M4: Taylor Series

(a) In class we computed a Taylor series for $\sin(x)$ centered at zero. Use the degree-seven Taylor polynomial to approximate $\sin(3) \approx T_7(3,0)$. (You don't need to numerically simplify this.)

Using the Taylor series remainder, find an upper bound for the error in this approximation.

Solution: We know that

$$\begin{aligned}
\sin(x) &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
T_7(x,0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
T_7(x,3) &= 3 - \frac{27}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} = 3 - \frac{37}{6} + \frac{243}{120} - \frac{2187}{5040} \\
&= 3 - \frac{9}{2} + \frac{81}{40} - \frac{243}{560} = \frac{51}{560} \approx 0.091.
\end{aligned}$$

We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$\begin{aligned}
|R_n(x)| &= \left| \frac{f^{(n+1)(z)}}{(n+1)!} x^{n+1} \right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_7(x)| &\leq \frac{x^{7+1}}{(7+1)!} \\
|R_7(3)| &\leq \frac{3^8}{8!} = \frac{729}{4480} \approx 0.16.
\end{aligned}$$

It would also be okay to observe that the eighth term is zero, so we could actually compute

$$\begin{aligned}
|R_n(x)| &= \left| \frac{f^{(n+1)(z)}}{(n+1)!} x^{n+1} \right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_8(x)| &\leq \frac{x^{8+1}}{(8+1)!} \\
|R_8(3)| &\leq \frac{3^9}{9!} = \frac{243}{4480} \approx 0.054.
\end{aligned}$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $x^3 e^{(x^5/4)}$, and then write down the first four non-zero terms of this series.

Solution:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\
e^{x^5/4} &= \sum_{n=0}^{\infty} \frac{1}{n!}(x^5/4)^n = \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n} \\
x^3 e^{x^5/4} &= \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n+3}
\end{aligned}
$$

The first four non-zero terms are

$$x^3 + \frac{1}{4}x^8 + \frac{1}{32}x^{13} + \frac{1}{6 \cdot 64}x^{18}.$$

(Note: this is *not* T_3 or T_4. It's T_{18}!)

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $(1-2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \binom{-3}{n} (-2x)^n = \sum_{n=0}^{\infty} \binom{-3}{n} (-2)^n x^n \\
T_4(x,0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4 \\
&= 1 + 6x + 24x^2 + 80x^3 + 240x^4
\end{aligned}
$$

S9: Applications of Taylor Series

(a) Use a degree-three Taylor polynomial to estimate $\sqrt{1.2}$.

Solution:

$$
\begin{aligned}
\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \\
\sqrt{1.2} &= 1 + \frac{.2}{2} - \frac{.04}{8} + \frac{.008}{16} = 1 + .1 - .005 + .0005 = 1.0955.
\end{aligned}
$$

(b) Use a Taylor series to compute $\lim_{x \to 0} \frac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{xe^{x^3}-x-x^4}{x^7} &= \lim_{x\to 0}\frac{(x+x^4+x^7/2+x^{10}/3!+\dots)-x+x^4}{x^7}\\ &= \lim_{x\to 0}\frac{x^7/2+x^{10}/3!+\dots}{x^7}\\ &= \lim_{x\to 0}\frac{1}{2}-\frac{x^3}{3!}+\cdots=\frac{1}{2}.\end{aligned}$$

(c) Using series, compute $\int_0^\pi 2x\cos(x^5)\,dx$.

Solution:

$$\begin{aligned}\cos(x) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^{2n}\\ \cos(x^5) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^{10n}\\ 2x\cos(x^5) &= \sum_{n=0}^\infty \frac{2(-1)^n}{(2n)!}x^{10n+1}\\ \int 2x\cos(x^5)\,dx &= \sum_{n=0}^\infty \frac{2(-1)^n}{(2n)!(10n+2)}x^{10n+2}+C\\ \int_0^\pi 2x\cos(x^5)\,dx &= \sum_{n=0}^\infty \frac{2(-1)^n}{(2n)!(10n+2)}\pi^{10n+2}\end{aligned}$$