Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 13
Due Monday, December 2

This week's mastery quiz has three topics. Everyone should submit S10. If you have a 4/4 on M4 or a 2/2 on S9, you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series
- Secondary Topic 10: Parametrization and Coordinates

Name:

Recitation Section:

M4: Taylor Series

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $(1+3x)^{2/3}$, and then write down the degree-three polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$f(x) = \sum_{n=0}^{\infty} \binom{2/3}{n} (3x)^n = \sum_{n=0}^{\infty} \binom{2/3}{n} (3)^n x^n$$

$$T_3(x,0) = 1 + \frac{2/3}{1} \cdot 3x + \frac{(2/3)(-1/3)}{2} \cdot 3^2 x^2 + \frac{(2/3)(-1/3)(-4/3)}{6} \cdot 3^3 x^3$$

$$= 1 + 2x - x^2 + \frac{4}{3}x^3.$$

(b) Find an upper bound for the error if you use $T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ to approximate $g(x) = \ln(1+x)$ at $x = .5$.

Solution: We can use the remainder theorem. We have

$$|R_3(x)| = \left| \frac{f^{(4)}(z)}{4!} x^4 \right| = \left| \frac{3!}{(1+z)^4 4!} x^4 \right| = \left| \frac{x^4}{4(1+z)^4} \right|$$

We know that z is between 0 and .5, so we have

$$\frac{1}{(1+z)^4} \leq \frac{1}{1^4} = 1$$

$$|R_3(.5)| = \left| \frac{.5^4}{4(1+z)^4} \right| \leq \frac{.5^4}{4} \cdot 1 = \frac{1}{64}.$$

(c) Let $f(x) = \sin(x)$. Use *the definition of a Taylor series* to find $T_3(x, \pi/3)$ (centered at $\pi/3$) for this function. (That is, find the terms up through the degree-three term.)

Solution:

$$f(x) = \sin(x) \qquad f(\pi/3) = \sqrt{3}/2$$

$$f'(x) = \cos(x) \qquad f'(\pi/3) = 1/2$$

$$f''(x) = -\sin(x) \qquad f''(\pi/3) = -\sqrt{3}/2$$

$$f'''(x) = -\cos(x) \qquad f'''(\pi/3) = -1/2$$

So we have

$$T_3(x, \pi/3) = \sqrt{3}/2 + \frac{1}{2}(x - \pi/3) - \frac{\sqrt{3}}{4}(x - \pi/3)^2 - \frac{1}{12}(x - \pi/3)^3$$

S9: Applications of Taylor Series

(a) Use a Taylor series to compute $\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} &= \lim_{x\to 0} \frac{(1 - x^4/2 + x^8/4! - x^{12}/6! + \dots) - 1 + x^4/2}{x^8} \\ &= \lim_{x\to 0} \frac{x^8/4! - x^{12}/6! + \dots}{x^8} \\ &= \lim_{x\to 0} \frac{1}{4!} - \frac{x^4}{6!} + \dots = \frac{1}{24}.\end{aligned}$$

(b) Use a degree-five Taylor polynomial to estimate arctan(.1).

Solution: We have

$$\arctan(x) \approx x - x^3/3 + x^5/5$$
$$\arctan(.1) \approx .1 - (.1)^3/3 + (.1)^5/5 = .1 - .00033\dots + .000002 = .09966866\dots.$$

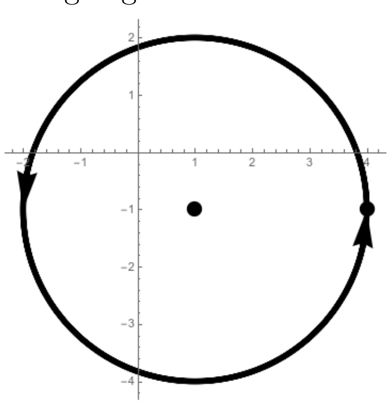
(c) If $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{n!+1} x^n$, compute $\int_3^5 f(x)$.

Solution:

$$\int f(x) = \sum_{n=0}^{\infty} \frac{1}{n!+1} x^{n+1} + C$$
$$\int_3^5 f(x) = \sum_{n=0}^{\infty} \frac{1}{n!+1} (5^{n+1} - 3^{n+1}).$$

S10: Parametrization

(a) Find a parametrization for the circle of radius 3 centered at $(1, -1)$, starting at $(4, -1)$ and going **counterclockwise twice** around the circle.



Solution: A circle has parametrization $\vec{r}(t) = (\cos(t), \sin(t))$. To make it radius 3 we multiply by 3, and then we shift it over to have center $(1, -1)$, and get

$$\vec{r}(t) = 3\cos(t) + 1, 3\sin(t) - 1.$$

In order to make it go around twice, we have $0 \leq t \leq 4\pi$. Alternatively, we could have $0 \leq 2 \leq 2\pi$ and use the equations

$$\vec{r}(t) = 3\cos(2t) + 1, 3\sin(2t) - 1.$$

(There are a bunch of other options that also work but these are the two most obvious to me.)

(b) Find the length of the curve parametrized by $x = e^t - t, y = 4e^{t/2}$ for $0 \leq t \leq 2$.

Solution: We have $x'(t) = e^t - 1$ and $y'(t) = 2e^{t/2}$, so the arc length is

$$\begin{aligned}
L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt \\
&= \int_0^2 \sqrt{e^{2t} - 2e^2 + 1 + 4e^t}\, dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, dt \\
&= \int_0^2 e^t + 1\, dt = e^t + t\Big|_0^2 = e^2 + 2 - 1 = e^2 + 1.
\end{aligned}$$

(c) Find the area inside the cardioid $r = 1 + \cos(\theta)$.

Solution: We have the polar area formula

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2} r^2\, d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(1 + \cos(\theta))^2\, d\theta \\
&= \int_0^{2\pi} \frac{1}{2} + \cos(\theta) + \frac{1}{2}\cos^2(\theta)\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta \\
&= \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} \\
&= \frac{1}{2}\left((2\pi + 0 + \pi + 0) - (0 + 0 + 0 + 0)\right) = \frac{3\pi}{2}.
\end{aligned}$$