Math 1232: Single-Variable Calculus 2

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Recitation 3

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Problem 1. Consider the integral $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$.

(a) We're going to have to do a u-substitution here. What u looks like it should work?

(b) What do we need to change the bounds to when we do the u-substitution?

(c) Compute $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$.

(d) Now try computing $\int \frac{1}{x\sqrt{\ln x}}\, dx$ to get the antiderivative.

(e) Now plug e^4 and e in to your antiderivative. What do you notice? How is this related to part (c)?

Solution:

(a) We take $u = \ln(x)$, and $du = \frac{dx}{x}$. This seems plausible because $\ln(x)$ is on the inside of a function.

(b) $\ln(e) = 1$ and $\ln(e^4) = 4$, so we have to integrate from 1 to 4.

(c)

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx = \int_1^4 \frac{1}{\sqrt{u}}\, du = 2\sqrt{u}\Big|_1^4 = 4 - 2 = 2.$$

(d)

$$\int \frac{1}{x\sqrt{\ln x}} dx = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u} + C = 2\sqrt{\ln(x)} + C.$$

(e) We get $2\sqrt{\ln(e^4)} = 2\sqrt{4} = 4$ and $2\sqrt{\ln(e)} = 2\sqrt{1} = 2$, which are the same numbers we got before. And specifically, you see we get 4 and 1 as intermediate answers here—these are the same numbers we got by doing the change of bounds.

Problem 2. Compute the following integrals.

(a) $\int \frac{e^x}{1+e^x}\,dx$.

(b) $\int \frac{\ln(x)}{x}\,dx$.

Solution:

(a) Take $u = 1 + e^x$ so $dx = \frac{du}{e^x}$. Then

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{e^x}{u}\frac{du}{e^x} = \int \frac{du}{u} = \ln|u| + C = \ln|1+e^x| + C.$$

(b) This one looks tricky, and you might have to mess around with it a bit to see, and try different things. But if we take $u = \ln(x)$ so that $dx = x\,du$, we see this is

$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln|x|)^2}{2} + C.$$

Problem 3 (Challenge)**.** Compute $\displaystyle\int \frac{dx}{1+e^x}$.

Solution: This problem becomes much easier if we multiply the top and bottom by e^{-x}. Then we have $\displaystyle\int \frac{e^{-x}}{e^{-x}+1}\,dx$. Set $u = e^{-x}$ so that $du = -e^{-x}\,dx$ and we have

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{du}{1+u} = -\ln(1+u) = -\ln(1+e^{-x}).$$

Alternatively, we can take $u = e^x, du = e^x\,dx$, and have

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(u+1)}.$$

Again nonobviously, we write

$$\begin{aligned}\int \frac{du}{u(u+1)} = \int \frac{1+u-u}{u(u+1)}\,du &= \int \frac{1+u}{u(u+1)}\,du - \int \frac{u}{u(u+1)}\,du \\ &= \int \frac{du}{u} - \int \frac{du}{u+1} \\ &= \ln(u) - \ln(u+1) = \ln(e^x) - \ln(e^x+1).\end{aligned}$$

Using properties of logs, you can check that this is the same as the previous answer. Or, if you prefer, you can write $x - \ln(e^x + 1)$.

Problem 4. (a) Compute $\sin(\arctan(5))$.

(b) Compute $\frac{d}{dx} \arccos(\sqrt{x})$

(c) Compute $\frac{d}{dx} \arctan(x + \sec(x))$

Solution:

(a) Our implicit triangle has side lengths of 5, 1, $\sqrt{26}$. So $\sin(\arctan(5)) = \frac{5}{\sqrt{26}}$.

(b)
$$\frac{d}{dx} \arccos(\sqrt{x}) = \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}.$$

(c)
$$\frac{d}{dx} \arctan(x + \sec(x)) = \frac{1}{1 + (x + \sec(x))^2} \cdot (1 + \sec(x)\tan(x)).$$

Problem 5. Compute the following integrals:

(a) $\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx.$

(b) $\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx.$

Solution:

(a) Take $u = \arcsin(x)$, and $du = \frac{dx}{\sqrt{1-x^2}}$. Then
$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\arcsin(x))^2 + C.$$

(b) Set $u = e^{2x}$ so $du = 2e^{2x}dx$. $g(0) = 1$ and $g(1) = e^2$. Then
$$\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx = \int_1^{e^2} \frac{1}{2(1+u^2)}\,du = \frac{1}{2}\arctan(u)\Big|_1^{e^2} = \frac{1}{2}\left(\arctan(e^2) - \arctan(1)\right).$$