Math 1232: Single-Variable Calculus 2

George Washington University Fall 2024

Recitation 4

Jay Daigle

September 20, 2024

Problem 1. (a) In class, we saw that $\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0$. What is $\lim_{x\to+\infty} \frac{\ln(x^2)}{x}$?

(b) Compute $\lim_{x\to+\infty} \frac{\ln(x^n)}{x}$ for $n > 0$.

(c) Compute $\lim_{x\to+\infty} \frac{\ln(x)}{x^\epsilon}$ for $\varepsilon > 0$.

(d) What do parts (a-c) tell you about the relationship between polynomials and $\ln(x)$?

Solution:

(a)

$$\lim_{x\to+\infty} \frac{\ln(x^2)}{x} = \lim_{x\to+\infty} \frac{2\ln(x)}{x} = 2 \cdot 0 = 0$$

since we know this limit from class.

Alternatively

$$\lim_{x\to+\infty} \frac{\ln(x^2)^{\nearrow\infty}}{x_{\searrow\infty}} \stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{2x/x^2}{1} = \lim_{x\to+\infty} \frac{2^{\nearrow 2}}{x_{\searrow\infty}} = 0.$$

(b)

$$\lim_{x\to+\infty} \frac{\ln(x^n)}{x} = \lim_{x\to+\infty} \frac{n\ln(x)}{x} = n \cdot 0 = 0$$

$$\begin{aligned}\lim_{x\to+\infty} \frac{\ln(x)^{\nearrow\infty}}{x^{\varepsilon}{}_{\searrow\infty}} &\stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{1/x}{\varepsilon x^{\varepsilon-1}} \\ &= \lim_{x\to+\infty} \frac{1^{\nearrow 1}}{\varepsilon x^{\varepsilon}{}_{\searrow\infty}} = 0.\end{aligned}$$

We see that $\ln(x)$ is *much much* smaller than any polynomial, when x is large. It doesn't matter how large a power we raise the inside of the logarithm to, or how small a power we raise the denominator to; in the limit, the logarithm will be infinitely smaller.

(a) In class we saw that $\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$. Compute $\lim_{x\to+\infty} \frac{e^x}{x^2}$.

(b) Compute $\lim_{x\to+\infty} \frac{e^x}{x^n}$ for $n > 0$.

(c) What do parts (e-f) tell you about the relationship between e^x and polynomials?

Solution:

(a)

$$\lim_{x\to+\infty} \frac{e^x{}_{\nearrow\infty}}{x^2{}_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty} \frac{e^x{}_{\nearrow\infty}}{2x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to\infty} \frac{e^x{}_{\nearrow\infty}}{2_{\searrow 2}} = +\infty.$$

(b) To work this out formally we'd need a "proof by induction", but we can see what's happening.

$$\begin{aligned}
\lim_{x\to+\infty} \frac{e^x{}_{\nearrow\infty}}{x^n{}_{\searrow\infty}} &=^{\text{L'H}} \frac{e^x{}_{\nearrow\infty}}{nx^{n-1}{}_{\searrow\infty}} \\
&=^{\text{L'H}} \frac{e^x{}_{\nearrow\infty}}{n(n-1)x^{n-2}{}_{\searrow\infty}} \\
&\vdots \qquad \vdots \\
&=^{\text{L'H}} \frac{e^x{}_{\nearrow\infty}}{n(n-1)(n-2)\ldots(3)(2)x_{\searrow\infty}} \\
&=^{\text{L'H}} \frac{e^x{}_{\nearrow\infty}}{n(n-1)(n-2)\ldots(3)(2)_{\searrow n(n-1)\ldots(3)(2)}} = +\infty.
\end{aligned}$$

(c) This is the converse of the logarithm. e^x is much *bigger* than x^n for any positive n, when x is large; so e^x is asymptotically bigger than any polynomial.

Problem 2. (a) We want to compute $\lim_{x\to\pi/2} \sec(x) - \tan(x)$.

(b) Can we use L'Hospital's Rule on this as written? Can we change it to a form where L'Hospital's Rule works?

(c) What is the limit?

Solution:

(a) This is a $\infty - \infty$ limit.

(b) We can't use L'Hospital's Rule because this isn't a fraction. But we can write

$$\lim_{x\to\pi/2} \sec(x) - \tan(x) = \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right)$$

(c)

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\
&= \lim_{x\to\pi/2} \frac{1 - \sin(x)^{\nearrow 0}}{\cos(x)_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to\pi/2} \frac{-\cos(x)^{\nearrow 0}}{-\sin(x)_{\searrow 1}} = \frac{0}{1} = 0.
\end{aligned}$$

Problem 3. Let's compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$

(a) What indeterminate form is this?

(b) If $y = x^{\frac{1}{\ln(x)-1}}$, what is $\ln|y|$?

(c) Compute $\lim_{x\to 0^+} \ln|y|$.

(d) Compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$.

Solution:

(a) This is 0^0.

(b) $\ln(y) = \frac{1}{\ln(x)-1} \ln(x)$.

(c)

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(x)^{\nearrow\infty}}{\ln(x) - 1_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to 0^+} \frac{1/x}{1/x} = 1$$

(d) $\lim_{x\to 0^+} y = e^1 = e$.

Problem 4. (a) We want to compute $\int x^2 e^{-3x}\, dx$. Why do we want to use integration by parts? What should be our u and dv, and why?

(b) Compute the integral.

(c) Now we want to compute $\int \cos(3x)e^{2x}\,dx$. Why do we want to use integration by parts? What should be our u and dv, and why? When we need to make another choice, what forces us to make that choice?

(d) Compute the integral.

Solution:

(a) We see a product of two functions that don't really have any useful interactions, so we want to use integration by parts. x^2 gets simpler when we differentiate, and e^{-3x} doesn't, so we're going to want to differentiate the x^2 term. So we set $u = x^2$ and $dv = e^{-3x}\,dx$, so that $du = 2x$ and $v = \frac{-1}{3}e^{-3x}$.

(b)

$$\begin{aligned}
\int x^2 e^{-3x}\,dx &= x^2\frac{-1}{3}e^{-3x} - \int 2x\frac{-1}{3}e^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(x\frac{-1}{3}e^{-3x} - \int \frac{-1}{3}e^{-3x}\,dx\right) \\
&= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C.
\end{aligned}$$

(c) We again see a product of two functions that don't have any useful interactions. Here it's really hard to see which one would be better to differentiate or integrate! I'm going to integrate the cosine and differentiate the e^{2x} but it doesn't matter that much. So I take $u = e^{2x}$ and $dv = \cos(3x)\,dx$, which gives $du = 2e^{2x}\,dx$ and $v = \frac{1}{3}\sin(3x)$.

(d)

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} - \int \frac{1}{3}\sin(3x)\cdot 2e^{2x}\,dx \\
&= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\int \sin(3x)e^{2x}\,dx.
\end{aligned}$$

Here we need to do integration by parts again, but we *have to* take e^{2x} to be our u this time, since we did last time. That means $dv = \sin(3x)\,dx$ and $v = \frac{-1}{3}\cos(3x)$. And we get

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\int \sin(3x)e^{2x}\,dx \\
&= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\left(\frac{-1}{3}\cos(3x)e^{2x} - \int \frac{-1}{3}\cos(3x)\cdot 2e^{2x}\,dx\right) \\
&= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} - \frac{4}{9}\int \cos(3x)e^{2x}\,dx.
\end{aligned}$$

Now we need to use our big trick, because this integral is the same one we started with. So we get

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} - \frac{4}{9}\int \cos(3x)e^{2x}\,dx \\
\frac{13}{9}\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} + C \\
\int \cos(3x)e^{2x}\,dx &= \frac{9}{13}\left(\frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x}\right) + C \\
&= \frac{3}{13}\sin(3x)e^{2x} + \frac{2}{13}\cos(3x)e^{2x} + C.
\end{aligned}$$

Problem 5. Compute $\int \arctan(x)\,dx$.

Solution: We know the derivative of arctan but not the integral. But integration by parts lets us convert! We can take $u = \arctan(x)$ and $dv = dx$, so then $du = \frac{dx}{1+x^2}$ and $v = x$. Then we have

$$\begin{aligned}
\int \arctan(x)\,dx &= x\arctan(x) - \int x\frac{dx}{1+x^2} \\
&= x\arctan(x) - \int \frac{x}{u}\cdot\frac{du}{2x} && (u = 1+x^2) \\
&= x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du \\
&= x\arctan(x) - \ln|u| + C = x\arctan(x) - \ln|x^2+1| + C.
\end{aligned}$$

(Again we see the close relationship between inverse trig and logarithms.)