Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 5
Due Monday, September 30

This week's mastery quiz has three topics. Everyone should submit M2. If you already have a 4/4 on M1, or a 2/2 on S2, you don't need to submit those again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 2: L'Hospital's Rule

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) $\displaystyle\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx =$

Solution: We can take $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{-u}{1+u^2}\,du$$

Then we can set $v = 1+u^2$ so that $dv = 2u\,du$ and we get

$$\begin{aligned}\int \frac{-u}{1+u^2}\,du = \int \frac{-1}{2}\frac{1}{v}\,dv &= \frac{-1}{2}\ln|v| + C \\ &= \frac{-1}{2}\ln|1+u^2| + C = \frac{-1}{2}\ln|1+\cos^2(x)| + C.\end{aligned}$$

(b) Compute $\displaystyle\frac{d}{dx}\left(\sqrt{x+1}\right)^x$

Solution:

$$\begin{aligned} y &= \sqrt{x+1}^x \\ ln|y| &= x\ln(\sqrt{x+1}) = \frac{1}{2}x\ln(x+1) \\ y'/y &= \frac{1}{2}\left(\ln(x+1) + \frac{x}{x+1}\right) \\ y' &= \frac{1}{2}\sqrt{x+1}^x\left(\ln(x+1) + \frac{x}{x+1}\right)\end{aligned}$$

(c) (Note this is a definite integral)
$\displaystyle\int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx =$

Solution: We can take $u = e^{2x}$ so $du = 2e^{2x}\,dx$ and

$$\begin{aligned}\int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx = \int_1^{e^4} \frac{1}{2}\frac{1}{u^2+1}\,du &= \frac{1}{2}\arctan(u)\Big|_1^{e^4} \\ &= \frac{1}{2}\arctan(e^4) - \frac{1}{2}\arctan(1) = \frac{1}{2}\arctan(e^4) - \frac{\pi}{8}.\end{aligned}$$

M2: Advanced Integration Techniques

(a) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx =$

Solution: We're going to set $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Then

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int \frac{\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)}\cos(\theta)\,d\theta \\ &= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\ &= \int \cot^2(\theta)\,d\theta \\ &= \int \csc^2(\theta) - 1\,d\theta \\ &= -\cot(\theta) - \theta + C.\end{aligned}$$

But we know that $\sin(\theta) = x$, so $x = \arcsin(\theta)$ and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sqrt{1-x^2}}{x}$. Then we get

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = -\frac{\sqrt{1-x^2}}{x} - \arcsin(x) + C.$$

(b) $\displaystyle\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt =$

Solution: We're going to take $u = \sec(2t)$ so that $du = 2\sec(2t)\tan(2t)\,dt$. We compute $u(0) = 1$ and $u(\pi/6) = \sec(\pi/3) = 2$. Then

$$\begin{aligned}\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt &= \int_1^2 \frac{1}{2}u^2\,du \\ &= \frac{u^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.\end{aligned}$$

(c) $\displaystyle\int \sin(2x)\cos(3x)\,dx =$

(Please do *not* use any product-of-trig-function identities we haven't discussed in class.)

Solution:

$$\begin{aligned}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) - \int \frac{2}{3}\cos(2x)\sin(3x)\,dx \\ \int \cos(2x)\sin(3x)\,dx &= -\frac{1}{3}\cos(2x)\cos(3x) - \int \frac{2}{3}\sin(2x)\cos(3x) \\ \int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\,dx \\ \frac{5}{9}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\big(+C\big) \\ \int \sin(2x)\cos(3x)\,dx &= \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.\end{aligned}$$

S2: L'Hospital's Rule

(a) $\lim_{x\to 2} \dfrac{e^{(x^2-4)} - x + 1}{x - 2} =$

Solution: The limit of the top and bottom are both 0, we can use L'Hospital's rule.

$$\lim_{x\to 2} \frac{e^{x^2-4} - x + 1^{\nearrow 0}}{x - 2_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 2} \frac{2xe^{x^2-4} - 1}{1} = 3.$$

(b) $\lim_{x\to\infty} x^{\frac{3}{2+\ln(x)}} =$

Solution:

$$\ln y = \frac{3}{2 + \ln(x)} \ln(x)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{3\ln(x)^{\nearrow\infty}}{2 + \ln(x)_{\searrow\infty}}$$

$$=^{\text{L'H}} \lim_{x\to\infty} \frac{3/x}{1/x} = 3$$

and thus

$$\lim_{x\to\infty} y = e^3.$$

(c) $\lim_{x\to+\infty} \dfrac{\arctan(x)}{\arctan(x) + 1} =$

Solution: $\lim_{x\to+\infty} \arctan(x) = \pi/2$, so this limit is $\frac{\pi/2}{\pi/2+1} \approx .611$.

Note: you cannot use L'Hospital's rule here! If you tried, you would get

$$\lim_{x\to+\infty} \frac{1/(x^2+1)}{1/(x^2+1)} = \lim_{x\to+\infty} 1 = 1$$

but that is not in fact the limit.