Math 1232: Single-Variable Calculus 2

George Washington University   Fall 2024

Recitation 5

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September 27, 2023

Problem 1. Compute $\int \sin^6(x)\,dx$.

Solution: By the double angle formula, we have

$$\begin{aligned}
\int \sin^6(x)\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int (1-\cos(2x))^3 \, dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\,dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + 3\frac{1+\cos(4x)}{2} - (1-\sin^2(2x))\cos(2x)\,dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + \frac{3}{2} + \frac{3}{2}\cos(4x) - \cos(2x) + \sin^2(2x)\cos(2x)\,dx \\
&= \frac{1}{8}\int \frac{5}{2} - 4\cos(2x) + \frac{3}{2}\cos(4x) + \sin^2(2x)\cos(2x)\,dx \\
&= \frac{1}{8}\left(\frac{5x}{2} - 2\sin(2x) + \frac{3}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\right) + C.
\end{aligned}$$

Problem 2. Compute $\displaystyle\int \sec^6(x)\tan^5(x)\,dx$ with two different approaches. Do you get the same answer either way?

Solution: One option is to reduce until we have two secant terms. Then we can set $u = \tan(x)$ and $du = \sec^2(x)\,dx$. We compute

$$\begin{aligned}\int \sec^6(x)\tan^5(x)\,dx &= \int \sec^2(x)(1+\tan^2(x))^2\tan^5(x)\,dx\\ &= \int \sec^2(x)\tan^5(x) + 2\sec^2(x)\tan^7(x) + \sec^2(x)\tan^9(x)\,dx\\ &= \int u^5 + 2u^7 + u^9\,du = \frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10} + C\\ &= \frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^8(x) + \frac{1}{10}\tan^{10}(x) + C.\end{aligned}$$

Alternatively, we could reduce until we have one tangent term, so we can set $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$. We compute

$$\begin{aligned}\int \sec^6(x)\tan^5(x)\,dx &= \int \sec^6(x)\tan(x)(\sec^2(x)-1)^2\,dx\\ &= \int \sec^{10}(x)\tan(x) - 2\sec^8(x)\tan(x) + \sec^6(x)\tan(x)\,dx\\ &= \int u^9 - 2u^7 + u^5\,du = \frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6 + C\\ &= \frac{1}{10}\sec^{10}(x) - \frac{1}{4}\sec^8(x) + \frac{1}{6}\sec^6(x) + C.\end{aligned}$$

These don't *look* the same, and they aren't—quite. They differ by $\frac{1}{60}$, but that's just a constant, so they are the same with the plus C. They're related by the identity that $\tan^2(x) + 1 = \sec^2(x)$, which is the identity we used to get these in the first place.

Problem 3 (Bonus)**.** Do one of the following two integrals. Explain why you don't want to do the other one.

(a) $\int \tan^2(x)\sec^3(x)\,dx$

(b) $\int \tan^3(x)\sec^3(x)\,dx$.

Solution: The second one is pretty straightforward. We can compute

$$\begin{aligned}\int \tan^3(x)\sec^3(x)\,dx &= \int \tan(x)\sec^5(x) - \tan(x)\sec^3(x)\,dx\\ &= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C.\end{aligned}$$

The first one, on the other hand, is extremely painful. You can't get it to a form with a useful u substitution, and will have to use integration by parts and other painful work. The answer turns out to be

$$\int \tan^2(x)\sec^3(x)\,dx = \frac{1}{32}\bigg(4\ln\left(\cos(x/2)-\sin(x/2)\right) - 4\ln\left(\cos(x/2)+\sin(x/2)\right) - \sec^4(x)\sin(3x) + 7\sec^3(x)\tan(x)\bigg).$$

Problem 4. Consider the integral $\int \frac{dx}{\sqrt{4x^2-1}}$.

(a) Which trig function would let us simplify that square root, and what identity are we using?

(b) What trigonometric substitution should we use here?

(c) Compute the antiderivative.

(d) Make sure to substitute your x back into the equation!

Solution:

(a) We want to use a $\sec(\theta)$ form, because we can use the identity that $\sec^2(\theta)-1 = \tan^2(\theta)$.

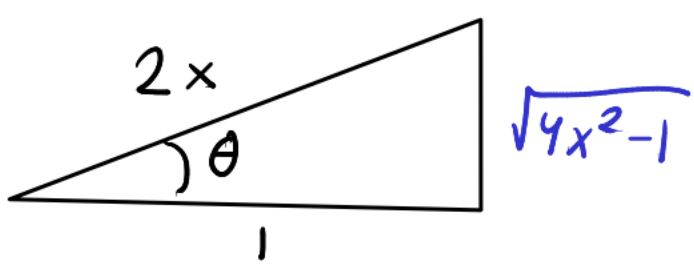
(b) Set $2x = \sec(\theta)$, so that $x = \frac{1}{2}\sec(\theta)$. Then $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$.

(c) We have

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}} \\ &= \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} \\ &= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

(d) We know that $\sec\theta = 2x$ by our definition of θ. To find $\tan\theta$ we draw a triangle: angle θ has hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan\theta = \sqrt{4x^2-1}$. Thus

$$\int \frac{dx}{\sqrt{4x^2-1}}\,dx = \ln|x+\sqrt{4x^2-1}/2| + C.$$



Problem 5. We want to find $\int \frac{x^5+x-1}{x^3+1}\,dx$.

(a) What's the first tool we need to apply here? (Hint: not partial fractions!)

(b) Once we get it in a more manageable form, things should simplify out nicely. What is the final integral?

Solution:

(a) Because the numerator is higher degree than the denominator, we need to start with a polynomial long division. We get
$$\frac{x^5+x-1}{x^3+1} = x^2 - \frac{x^2-x+1}{x^3+1} = x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}.$$

(b) We don't even need to do a partial fractions decomposition here: instead it just factors. We have

Thus
$$\begin{aligned}\int \frac{x^5+x-1}{x^3+1}\,dx &= \int x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}\,dx \\ &= \int x^2 - \frac{1}{x+1}\,dx = \frac{x^3}{3} - \ln|x+1|.\end{aligned}$$

Problem 6. We've looked briefly at the integral $\int \frac{1}{1+e^x}\,dx$. Let's try it again with our new tools.

(a) Try the substitution $u = e^x$. What do you get? What tools can apply to the result?

(b) Do a partial fractions decomposition to get the integral.

Solution:

(a) If $u = e^x$ then $du = e^x\,dx$.
$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+e^x}\frac{du}{e^x} = \int \frac{du}{u(1+u)}.$$

(b) This looks like a fraction with a factorable denominator. So a partial fractions decomposition gives us
$$\begin{aligned}\frac{1}{u(1+u)} &= \frac{A}{u} + \frac{B}{1+u} \\ 1 &= A(1+u) + B(u) = A + (A+B)u\end{aligned}$$

so $A = 1$ and $B = -1$.

(Alternatively, plugging in $u = 0$ gives $1 = A$, and plugging in $u = -1$ gives $1 = -B$.)

Thus

$$\begin{aligned}
\int \frac{1}{1+e^x}\,dx &= \int \frac{du}{u(1+u)} \\
&= \int \frac{1}{u} - \frac{1}{1+u}\,du \\
&= \ln|u| - \ln|1+u| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C.
\end{aligned}$$

Problem 7 (Bonus)**.** Let's see if we can work out the integral of secant. This isn't at all obvious!

(a) We want $\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$. Since this is a fraction, we can multiply the top and bottom through by $\cos(x)$. This makes the expression more complicated, but it does allow us to use a trig identity. What do we get?

(b) Now we can do a u substitution. What u substitution seems reasonable? Does it help us at all?

(c) Now we can use partial fractions to finish the problem off. We wind up with an awkward answer, but an answer.

(d) The most common formula for the integral of $\sec(x)$ is $\ln|\sec(x) + \tan(x)| + C$. Is that the same as what you got? (Hint: use logarithm laws and multiplication by the conjugate.)

Solution:

(a) We get $\int \frac{\cos(x)}{\cos^2(x)}\,dx$. The bottom allows us to use the pythagorean identity, and now we're trying to compute $\int \frac{\cos(x)}{1-\sin^2(x)}\,dx$.

(b) The only really reasonable choice here is $u = \sin(x)$, so $du = \cos(x)\,dx$. Then we have

$$\int \frac{\cos(x)}{1-\sin^2(x)}\,dx = \int \frac{1}{1-u^2}\,du.$$

(c) We write

$$\begin{aligned}
\frac{1}{1-u^2} &= \frac{A}{1-u} + \frac{B}{1+u} \\
1 &= A(1+u) + B(1-u).
\end{aligned}$$

Plugging in $u = 1$ gives us that $2B = 1$ and plugging in $u = -1$ gives us $2A = 1$, so we have

$$\begin{aligned}
\int \frac{\cos(x)}{1-\sin^2(x)}\,dx &= \int \frac{1}{1-u^2}\,du \\
&= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du \\
&= \frac{1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + C \\
&= \frac{1}{2}\left(-\ln|1-\sin(x)| + \ln|1+\sin(x)|\right) + C.
\end{aligned}$$

(d) It doesn't look the same, but it is!

$$\begin{aligned}
\frac{1}{2}\left(-\ln|1-\sin(x)| + \ln|1+\sin(x)|\right) &= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{1-\sin^2(x)}\right| \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{\cos^2(x)}\right| \\
&= \ln\left|\frac{1+\sin(x)}{\cos(x)}\right| \\
&= \ln\left|\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right| = \ln|\sec(x)+\tan(x)|.
\end{aligned}$$

Problem 8 (Bonus)**.** What if we want to find $\displaystyle\int \frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)}\,dx$?

Solution: The numerator is lower degree than the denominator, so we can begin a partial fraction decomposition immediately. We set up:

$$\begin{aligned}
\frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+1} + \frac{Dx+E}{x^2+4x+5} \\
x^4+6x^3+4x^2+8x+11 &= A(x-1)(2x+1)(x^2+4x+5) + B(2x+1)(x^2+4x+5) \\
&\quad + C(x-1)^2(x^2+4x+5) + (Dx+E)(x-1)^2(2x+1) \\
&= A(2x^4+7x^3+5x^2-9x-5) + B(2x^3+9x^2+14x+5) \\
&\quad + C(x^4+2x^3-2x^2-6x+5) + D(2x^4-3x^3+x) \\
&\quad + E(2x^3-3x^2+1) \\
&= (2A+C+2D)x^4 + (7A+2B+2C-3D+2E)x^3 \\
&\quad + (5A+9B-2C-3E)x^2 + (-9A+14B-6C+D)x \\
&\quad + (-5A+5B+5C+E)
\end{aligned}$$

We get a system of equations

$$\begin{aligned} 2A + C + 2D &= 1 & 7A + 2B + 2C - 3D + 2E &= 6 \\ 5A + 9B - 2C - 3E &= 4 & -9A + 14B - 6C + D &= 8 \\ -5A + 5B + 5C + E &= 11. \end{aligned}$$

After solving this (admittedly nasty) collection of equations, we see that $A = 0, B = 1, C = 1, D = 0, E = 1$. So

$$\begin{aligned} \int \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)}\, dx &= \int \frac{1}{(x+1)^2} + \frac{1}{2x+1} + \frac{1}{x^2+4x+5}\, dx \\ &= -(x+1)^{-1} + \frac{1}{2}\ln|2x+1| + \int \frac{dx}{x^2+4x+5}. \end{aligned}$$

We complete the square, and see that $x^2 + 4x + 5 = (x+2)^2 + 1$, so we use $u = x + 2$ and get

$$\int \frac{dx}{x^2+4x+5} = \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

Thus

$$\int \frac{x^4 + 6x^3 + 4x^2 + 8x + 11}{(x-1)^2(2x+1)(x^2+4x+5)}\, dx = -(x+1)^{-1} + \frac{1}{2}\ln|2x+1| + \arctan(x+2) + C.$$