Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 6
Due Monday, October 7

This week's mastery quiz has four topics. Everyone should submit M2 (even if you got a 2 last week!), S3, and S4. If you already have a 4/4 on M1, you don't need to submit it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) Compute $\frac{d}{dx}x^{e^x}$

Solution:

$$\begin{aligned}
y &= x^{e^x} \\
ln|y| &= e^x \ln|x| \\
y'/y &= e^x \ln|x| + \frac{e^x}{x} \\
y' &= e^x \ln|x| x^{e^x} + \frac{1}{x} e^x x^{e^x}.
\end{aligned}$$

(b) Compute $\int_0^{\ln(5)} e^x\sqrt{1+3e^x}\,dx$.

Solution: Set $u = 1 + 3e^x$, so $du = 3e^x\,dx$, and we see that $u(0) = 4$ and $u(\ln(5)) = 16$.

$$\begin{aligned}
\int_0^{\ln(5)} e^x\sqrt{1+3e^x}\,dx &= \int_4^{16} \frac{1}{3}\sqrt{u}\,du \\
&= \frac{2}{9}u^{3/2}\Big|_4^{16} = \frac{2}{9}\cdot 64 - \frac{2}{9}\cdot 8 = \frac{112}{9}.
\end{aligned}$$

Alternatively, we could substitute our integral back:

$$\begin{aligned}
\int e^x\sqrt{1+3e^x}\,dx &= \int \frac{1}{3}\sqrt{u}\,du \\
&= \frac{2}{9}u^{3/2} + C = \frac{2}{9}(1+3e^x)^{3/2} + C. \\
\int e^x\sqrt{1+3e^x}\,dx &= \frac{2}{9}(1+3e^x)^{3/2}\Big|_0^{\ln(5)} \\
&= \frac{2}{9}\cdot 64 - \frac{2}{9}\cdot 8 = \frac{112}{9}.
\end{aligned}$$

(c) Compute $\int \frac{\sin(2t)+\cos(2t)}{\sin(2t)-\cos(2t)}\,dt$.

Solution: We can take $u = \sin(2t) - \cos(2t)$. Then $du = (2\cos(2t) + 2\sin(2t))dt$, and we get

$$\begin{aligned}
\int \frac{\sin(2t)+\cos(2t)}{\sin(2t)-\cos(2t)}\,dt &= \int \frac{\sin(2t)+\cos(2t)}{u}\frac{du}{2\cos(2t)+2\sin(2t)} \\
&= \int \frac{1}{2}u\,du = \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|\sin(2t)-\cos(2t)| + C.
\end{aligned}$$

Name: Recitation Section:

M2: Advanced Integration Techniques

(a) Compute $\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\,dx$.

Solution: We need to do a partial fractions decomposition. We have

$$\begin{aligned}
\frac{4x^2 - x + 10}{(x-2)(x^2+4)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \\
4x^2 - x + 10 &= A(x^2+4) + (Bx+C)(x-2).2 : \quad 24 &= A \cdot 8 \\
A &= 30 : \quad 10 &= 4A + (C)(-2) = 12 - 2C \\
C &= 6 - 5 = 1 \\
1 : \quad 13 &= 3(1+4) + (B+1)(1-2) = 15 - B - 1 \\
B &= 1.
\end{aligned}$$

Thus we compute

$$\begin{aligned}
\int \frac{4x^2 - x + 10}{(x-2)(x^2+4)}\,dx &= \int \frac{3}{x-2} + \frac{x+1}{x^2+4}\,dx \\
&= \int \frac{3}{x-2} + \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx \\
&= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{4}\int \frac{1}{(x/2)^2+1}\,dx \\
&= 3\ln|x-2| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$

(b) $\int_{\pi/9}^{\pi/6} x\cos(3x)\,dx =$

Solution:

$$\begin{aligned}
\int_{\pi/9}^{\pi/6} x\cos(3x)\,dx &= \frac{1}{3}x\sin(3x)\Big|_{\pi/9}^{\pi/6} - \int_{\pi/9}^{\pi/6} \frac{1}{3}\sin(3x)\,dx \\
&= \frac{\pi}{18}\sin(\pi/2) - \frac{\pi}{27}\sin(\pi/3) + \frac{1}{9}\cos(3x)\Big|_{\pi/9}^{\pi/6} \\
&= \frac{\pi}{18} - \frac{\pi\sqrt{3}}{54} + \frac{1}{9}\cos(\pi/2) - \frac{1}{9}\cos(\pi/3) \\
&= \frac{\pi}{18} - \frac{\pi\sqrt{3}}{54} - \frac{1}{18}.
\end{aligned}$$

(c) $\int (x-1)^3\sqrt{2x - x^2}\,dx =$

Solution: We're going to set $x-1=\sin\theta$ so that $dx=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int (x-1)^3\sqrt{2x-x^2}\,dx \int (x-1)^3\sqrt{1-(x-1)^2}\,dx \\ &= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta \\ &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta \\ &= \int \sin(\theta)\cos^2(\theta)-\sin(\theta)\cos^4(\theta)\,d\theta \\ &= -\frac{1}{3}\cos^3(\theta)+\frac{1}{5}\cos^5(\theta)+C.\end{aligned}$$

But

$$\cos(\theta)=\cos(\arcsin(x-1))=\sqrt{1-(x-1)^2}=\sqrt{2x-x^2},$$

so

$$\int (x-1)^3\sqrt{2x-x^2}\,dx=\frac{1}{5}\sqrt{2x-x^2}^5-\frac{1}{3}\sqrt{2x-x^2}^3+C.$$

S3: Numeric Integration

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_5^9 (x+4)^{3/2}\,dx$ to within 1/10? Use the trapezoid rule with that many intervals to approximate the integral.

(Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$\begin{aligned} f'(x)\frac{3}{2}(x+4)^{1/2} f''(x) &= \frac{3}{4}(x+4)^{-1/2}=\frac{3}{4\sqrt{x+4}} \\ f''(5) &= \frac{1}{4} \\ |E_M| &\leq \frac{1/4\cdot 4^3}{12\cdot n^2}\leq\frac{1}{10} \\ n^2 &\geq 40/3\approx 13.3 \\ n &\geq 4\end{aligned}$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$\begin{aligned}\int_5^9 (x+4)^{3/2}\,dx &\approx \frac{\sqrt{9}^3+\sqrt{10}^3}{2}+\frac{\sqrt{10}^3+\sqrt{11}^3}{2}+\frac{\sqrt{11}^3+\sqrt{12}^3}{2}+\frac{\sqrt{12}^3+\sqrt{13}^3}{2} \\ &\approx \frac{1}{2}9^{3/2}+10^{3/2}+11^{3/2}+12^{3/2}+\frac{1}{2}13^{3/2}.\end{aligned}$$

We can stop there, but numerically this is roughly 146.61. The true answer is approximately 146.54 so this is within the expected error bound.

(b) Suppose we have

$$g(0) = 5 \quad g(1) = 4 \quad g(2) = 7 \quad g(3) = 4 \quad g(4) = 2 \quad g(5) = 3 \quad g(6) = 5$$

Approximate $\int_0^6 g(x)\,dx$ using the Midpoint rule and using Simpson's rule.

Solution: For the midpoint rule, we have

$$M_3 = 2 \cdot g(1) + 2 \cdot g(3) + 2 \cdot g(5) = 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 3 = 22.$$

For Simpson's rule, we have

$$\begin{aligned} S_6 &= \frac{1}{3}\left(5 + 4 \cdot 4 + 2 \cdot 7 + 4 \cdot 4 + 2 \cdot 2 + 4 \cdot 3 + 5\right) \\ &= \frac{1}{3}(5 + 16 + 14 + 16 + 4 + 12 + 5) = \frac{1}{3} \cdot 72 = 24 \end{aligned}$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx$.

Solution: We know that $\frac{1}{\sqrt[3]{x^2}}$ is undefined at zero. So we need to split this in half:

$$\begin{aligned} \int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx &= \int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx \\ &= \lim_{t\to 0^-} \int_{-1}^{t} x^{-2/3}\,dx + \lim_{s\to 0^+} \int_{s}^{1} x^{-2/3}\,dx \\ &= \lim_{t\to 0^-} 3x^{1/3}\Big|_{-1}^{t} + \lim_{x\to 0^+} 3x^{1/3}\Big|_{s}^{1} \\ &= 3 \lim_{t\to 0^-} \left((\sqrt[3]{t} - \sqrt[3]{-1}) + (\sqrt[3]{1} - \sqrt[3]{s})\right) \\ &= 3(0 + 1 + 1 - 0) = 6. \end{aligned}$$

Compute $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

Solution: We'll take $u = -x^2$ so $du = -2x\,dx$ and $xe^{-x^2} = \frac{-1}{2}e^u$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_{-\infty}^{t} xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\lim_{s\to-\infty}\int_{s}^{t} xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\lim_{s\to-\infty}\int_{-s^2}^{-t^2} -\frac{1}{2}e^u\,du \\
&= \lim_{t\to\infty}\lim_{s\to\infty} -\frac{1}{2}e^u\Big|_{-s^2}^{-t^2} = \lim_{t\to\infty}\lim_{s\to\infty}\frac{1}{2}e^{-s^2} - \frac{1}{2}e^{-t^2} \\
&= 0 - 0 = 0.
\end{aligned}$$