Math 1232: Single-Variable Calculus 2
George Washington University Spring 2023
Recitation 6

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February 23, 2023

Problem 1. We want to find the cross-sectional area of a two-meter-long airplane wing. We measure its width every 20 centimeters, and get: 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, 2.8. Use the trapezoidal rule and Simpson's rule to estimate the area of the wing.

Solution:

$$\begin{aligned}
T_{10} &= .2\left(\frac{5.8+20.3}{2}+\frac{20.3+26.7}{2}+\frac{26.7+29.0}{2}+\frac{29.0+27.6}{2}+\frac{27.6+27.3}{2}\right.\\
&\quad\left.+\frac{27.3+23.8}{2}+\frac{23.8+20.5}{2}+\frac{20.5+15.1}{2}+\frac{15.1+8.7}{2}+\frac{8.7+2.8}{2}\right)\\
&= 40.66\\
S_{10} &= \frac{1}{15}(5.8+4\cdot 20.3+2\cdot 26.7+4\cdot 29.0+2\cdot 27.6+4\cdot 27.3\\
&\quad +2\cdot 23.8+4\cdot 20.5+2\cdot 15.1+4\cdot 8.7+2.8)\\
&= \frac{618.2}{15} = 41.2133.
\end{aligned}$$

Problem 2. Consider the function $f(x) = x^2 + 1$.

(a) Use the trapezoid rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(b) Use the midpoint rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(c) Use Simpson's rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(d) Which of these do you expect to be most accurate? Which do you expect to be least accurate?

(e) Compute $\int_{-4}^{2} f(x)\,dx$. What do you find? Why?

Solution:

(a)

$$\begin{aligned}\int_{-4}^{2} x^2+1\,dx &\approx \frac{f(-4)+f(-3)}{2}+\frac{f(-3)+f(-2)}{2}+\frac{f(-2)+f(-1)}{2}\\ &\quad+\frac{f(-1)+f(0)}{2}+\frac{f(0)+f(1)}{2}+\frac{f(1)+f(2)}{2}\\ &=\frac{17+10}{2}+\frac{10+5}{2}+\frac{5+2}{2}+\frac{2+1}{2}+\frac{1+2}{2}+\frac{2+5}{2}\\ &=\frac{1}{2}(27+15+7+3+3+7)=\frac{62}{2}=31.\end{aligned}$$

Alternatively, we could write

$$\begin{aligned}\int_{-4}^{2} x^2+1\,dx &\approx \frac{1}{2}\Big(f(-4)+2f(-3)+2f(-2)+2f(-1)+2f(0)+2f(1)+f(2)\Big)\\ &=\frac{1}{2}\Big(17+20+10+4+2+4+5\Big)=\frac{1}{2}\cdot 62=31.\end{aligned}$$

(b)

$$\begin{aligned}\int_{-4}^{2} x^2+1\,dx &\approx f(-3.5)+f(-2.5)+f(-1.5)+f(-0.5)+f(0.5)+f(1.5)\\ &=13.25+7.25+3.25+1.25+1.25+3.25=29.5.\end{aligned}$$

(c)

$$\begin{aligned}\int_{-4}^{2} x^2+1\,dx &\approx \frac{1}{3}\Big(f(-4)+4f(-3)+2f(-2)+4f(-1)+2f(0)+4f(1)+f(2)\Big)\\ &=\frac{1}{3}\Big(17+40+10+8+2+8+5\Big)=30.\end{aligned}$$

(d) We'd expect these to be roughly in increasing order of accuracy, with the trapezoid rule being the least accurate, and the midpoint rule being the most accurate.

(e)

$$\begin{aligned}\int_{-4}^{2} x^2+1\,dx &= \frac{x^3}{3}+x\Big|_{-4}^{2}\\ &=\frac{8}{3}+2-\left(\frac{-64}{3}-4\right)\\ &=\frac{72}{3}+6=24+6=30.\end{aligned}$$

Simpson's rule got it exactly correct! This makes sense, because the fourth derivative is zero.

Problem 3. Let $g(x) = e^{-x^2}$, and suppose we want to compute $\int_{-1}^{2} e^{-x^2}\,dx$, and get the answer correct to two decimal places.

(a) We can compute that $g''(x)$ varies between -2 and $.9$ when x is in $[-1, 2]$. What value should we take for K?

(b) How many subintervals should we use to get the answer correct to within two decimal places using the trapezoid rule?

(c) How many subintervals should we use to get the answer correct to within two decimal places using the midpoint rule?

(d) We can compute that $g''''(x)$ varies between -8 and 12. What value should we take for L?

(e) How many subintervals should we use to get the answer correct to within two decimal places using Simpson's rule?

Solution:

(a) Since $g''(x)$ varies between -2 and $.9$, the largest the absolute value can get will be 2. Thus we should take $K = 2$.

If we wanted to compute this for ourselves, we could observe that

$$\begin{aligned} g''(x) &= 2e^{-x^2}(2x^2 - 1) \\ g'''(x) &= -4xe^{-x^2}(2x^2 - 3). \end{aligned}$$

We want to maximize g'' so we look for the zeroes of g''', which happen at 0 and at $\pm\sqrt{3/2}$. We can ignore $-\sqrt{3/2}$ since it's not in $[-1, 2]$, but we do need to check the endpoints, so we compute

$$\begin{aligned} g''(-1) &\approx .74 & g''(0) &= -2 \\ g''(\sqrt{3/2}) &\approx .89 & g''(2) &\approx .26. \end{aligned}$$

Thus the absolute minimum is -2 and the absolute max is about .89.

(b) The error in the trapezoid rule is

$$\begin{aligned} |E_T| \leq \frac{2(3)^3}{12n^2} &< \frac{1}{100} \\ 5400 &< 12n^2 \\ 450 &< n^2 \\ 21.2 &< n \end{aligned}$$

so we'd need at least 22 intervals.

(c) The error in the midpoint rule is

$$\begin{aligned} |E_T| \leq \frac{2(3)^3}{24n^2} &< \frac{1}{100} \\ 5400 &< 24n^2 \\ 225 &< n^2 \\ 15 &< n \end{aligned}$$

so we'd need *sixteen* intervals to make sure the error was *less than* 1/100, but 15 will guarantee the error is *at most* 1/100.

(d) We need to take $L = 12$.

(e) The error in Simpson's rule is

$$\begin{aligned} |E_S| \leq \frac{12 \cdot 3^5}{180n^4} &< \frac{1}{100} \\ 291600 &< 180n^4 \\ 1620 &< n^4 \\ 6.344 &< n^4 \end{aligned}$$

so we'd only need five intervals.

Problem 4. We want to compute $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\, dx$.

(a) Can you compute an antiderivative? Can you evaluate it at 0 and 2?

(b) Did part (a) finish the problem? Sketch a picture of the graph. What should we be concerned about?

(c) Carefully set up a computation that will find $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\, dx$. (Hint: It should have two limit operations in it.)

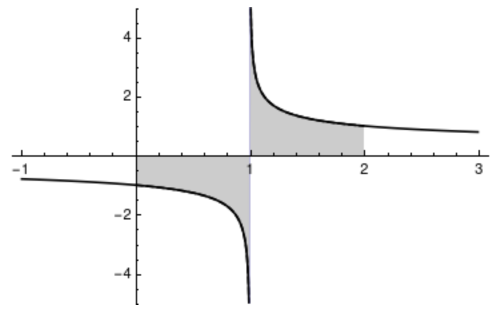
(d) What did we learn from this that we didn't learn from (a)?

Solution:

(a) We know that

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx = \frac{3}{2}(x-1)^{2/3}$$
$$\frac{3}{2}(x-1)^{2/3}\Big|_0^2 = \frac{3}{2}(1)^{2/3} - \frac{3}{2}(-1)^{2/3} = 0.$$

(b) This is not a correct way to compute this integral, because we're assuming everything makes sense in the middle. But it does not.



This is an improper integral, because the function is discontinuous and undefined at 1.

(c) To compute, we break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to 1^+}\int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-}\left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_0^t - \lim_{s\to 1^+}\left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_s^2 \\
&= \lim_{t\to 1^-}\left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\cdot 1\right) + \lim_{s\to 1^+}\left(\frac{3}{2}\cdot 1 - \frac{3}{2}(s-1)^{2/3}\right) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

(d) We get the "same answer" here, of 0. But this computation gives us an extra piece of information: all the area involved is finite. Saying the net area is $3/2 - 3/2$ is very different from trying to say it's $\infty - \infty$ somehow.

Problem 5 (Bonus)**.** We want to figure out if $\int_0^{+\infty} e^{-x^2}\,dx$ converges—that is, if it's finite or infinite.

(a) If we can find an antiderivative, we can just compute the improper integral directly. Why doesn't that work?

(b) Since we can't integrate this directly we might want to use a comparison test. We need to find an easy-to-integrate function that's larger than e^{-x^2}. Find a function $f(x)$ that makes $f(x)e^{-x^2}$ easy to integrate.

(c) If $f(x) \geq 1$, then we can just integrate $f(x)e^{-x^2}$. Is it?

(d) This is where we can pull in a trick. Is there some a where $f(x) > 1$ when $x > a$? (You may need to adjust your $f(x)$ here, especially the sign. It's fine as long as you can still integrate it.)

(e) We know $\int_a^{+\infty} e^{-x^2}\,dx \leq \int_a^{+\infty} f(x)e^{-x^2}\,dx$. Compute the new improper integral; is it finite?

(f) Now we just have to deal with $\int_0^a e^{-x^2}\,dx$. We can't do that integral exactly, but that's fine: you should be able to tell whether it's finite or not without doing any calculations. How?

(g) Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

Solution:

(a) This is precisely the standard function we know we don't have an elementary antiderivative for. So that won't help.

(b) The obvious value for $f(x)$ is $-2x$, since that's the chain rule from e^{-x^2}. It will work out easier in the long run if we take $f(x) = 2x$, though.

(c) Clearly $-2xe^{-x^2}$ is often less than e^{-x^2}, since the first is negative and the second is positive. Even if we fix that problem, we still see that $2xe^{-x^2} < e^{-x^2}$ when $x < 1/2$.

(d) If we take $a = 1/2$, or any larger number, we fix this problem. I'm going to take $a = 1$; then we have $2xe^{-x^2} > e^{-x^2}$ for $x > 1$.

(e) In class we showed that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges. In particular we showed that

$$\begin{aligned}\int_1^{+\infty} 2xe^{-x^2}\,dx &= \lim_{s\to+\infty}\int_1^s 2xe^{-x^2}\,dx\\ &= \lim_{s\to+\infty}\int_1^{s^2} e^{-u}\,du\\ &= \lim_{s\to+\infty} -e^{-u}\big|_1^{s^2}\\ &= \lim_{s\to+\infty} -e^{-s^2}-(-e^{-1}) = 0+\frac{1}{e}=\frac{1}{e}.\end{aligned}$$

So we know that

$$\int_1^{+\infty} e^{-x^2}\,dx < \int_1^{+\infty} 2xe^{-x^2}\,dx = \frac{1}{e}$$

and thus $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

(f) $\int_0^1 e^{-x^2}\,dx$ isn't an integral we can do. But it's a nice, proper integral, so nothing weird can happen. An integral of a function that's defined and continuous everywhere on the closed interval $[0,1]$ will always *converge*. (Numerically, it works out to about 0.75.)

(g) We have concluded that

$$\begin{aligned}\int_0^{+\infty} e^{-x^2}\,dx &= \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx\\ &< \int_0^1 e^{-x^2}\,dx + \frac{1}{e} < \infty.\end{aligned}$$

In fact, this tells us that $\int_0^{+\infty} e^{-x^2}\,dx < \frac{1}{e} + .75 \approx 1.12$.

Using fancy techniques from cmoplex analysis, we can determine that $\int_0^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \approx .88$. That's way outside the scope of what we can do in this course, though.