Math 1232 Fall 2024
Single-Variable Calculus 2 Section 11
Mastery Quiz 6
Due Monday, October 7

This week's mastery quiz has four topics. Everyone should submit S5. If you already have a 4/4 on M2, you don't need to submit it again, and if you already have a 2/2 on S3 or S4, you don't need to submit them again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please **don't discuss the actual quiz questions with other students in the course**.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\int_0^3 x^2 e^{2x}\,dx =$

Solution:

$$\begin{aligned}
\int_0^3 x^2 e^{2x}\,dx &= \frac{1}{2}x^2e^{2x}\Big|_0^3 - \int_0^3 xe^{2x}\,dx \\
&= \frac{9}{2}e^6 - \int_0^3 xe^{2x}\,dx \\
\int_0^3 xe^{2x}\,dx &= \frac{1}{2}xe^{2x}\Big|_0^3 - \int_0^3 \frac{1}{2}e^{2x}\,dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\Big|_0^3 \\
&= \frac{3}{2}e^6 - \frac{1}{4}e^6 + \frac{1}{4} \\
\int_0^3 x^2 e^{2x}\,dx &= \frac{9}{2}e^6 - \frac{3}{2}e^6 + \frac{1}{4}e^6 - \frac{1}{4} = \frac{13}{4}e^6 - \frac{1}{4}.
\end{aligned}$$

(b) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx =$

Solution:

$$\begin{aligned}
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\
x^2+x-4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\
2 &= -2B \Rightarrow B = -1 \\
-4 &= 4C \Rightarrow C = -1 \\
-4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\
\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx \\
&= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C.
\end{aligned}$$

(c) $\int \frac{\sqrt{4x^2-1}}{x}\,dx =$

Solution: We set $2x = \sec(\theta)$, so $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{4x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta - 1}}{\frac{1}{2}\sec\theta}\frac{1}{2}\sec(\theta)\tan\theta\,d\theta \\ &= \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta \\ &= \tan(\theta) - \theta + C\end{aligned}$$

Then we know $\sec\theta = 2x$ so we can make a triangle with hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan(\theta) = \sqrt{4x^2-1}$. Then we can say either $\theta = \operatorname{arcsec}(2x)$ or $\theta = \arctan(\sqrt{4x^2-1})$, and we have

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \sqrt{4x^2-1} - \arctan(\sqrt{4x^2-1}) + C = \sqrt{4x^2-1} - \operatorname{arcsec}(2x) + C.$$

S3: Numeric Integration

(a) Let $f(x) = x^3 + x$. How many intervals do you need with the midpoint rule to approximate $\int_1^2 x^3 + x\,dx$ to within 1/10? Compute the integral with that many integrals. (Feel free to use a calculator to plug values into f, but show every step.)

Solution: We have

$$\begin{aligned}f''(x) &= 6x \\ f'(2) &= 12 \\ |E_M| &\leq \frac{12 \cdot 1^3}{24 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 5 \\ n &> 2\end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\int_1^2 x^3 + x\,dx \approx \frac{1}{3}f(7/6) + \frac{1}{3}f(9/6) + \frac{1}{3}f(11/6) \approx \frac{1}{3}(2.75 + 4.875 + 8.00) = \frac{1}{3}15.625 \approx 5.21.$$

(Since the true answer is 5.25 this is in fact within our error bound.)

(b) Suppose we have

$$g(0) = 2.4 \qquad g(1) = 4 \qquad g(2) = 2.7 \qquad g(3) = 2.3 \qquad g(4) = 1.7$$

Approximate $\int_0^4 g(x)\,dx$ using the Trapezoid rule, and then using Simpson's rule.

Solution: For the trapezoid rule, we have

$$\begin{aligned}T_4 &= 1\cdot\frac{2.4+4}{2}+1\cdot\frac{4+2.7}{2}+1\cdot\frac{2.7+2.3}{2}+1\cdot\frac{2.3+1.7}{2}\\ &= \frac{1}{2}(6.4+6.7+5+4.0)=\frac{1}{2}\cdot 22.1 = 11.05.\end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned}S_4 &= \frac{1}{3}\left(2.4+4\cdot 4+2\cdot 2.7+4\cdot 2.3+1.7\right)\\ &= \frac{1}{3}(2.4+12+5.4+9.2+1.9)=\frac{1}{3}\cdot 34.7\approx 11.5667.\end{aligned}$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^3 \frac{x}{x^2-1}\,dx =$

Solution:

$$\begin{aligned}\int_1^3 \frac{x}{x^2-1}\,dx &= \lim_{t\to 1^+}\int_t^3 \frac{x}{x^2-1}\,dx\\ &= \lim_{t\to 1^+}\frac{1}{2}\ln|x^2-1|\Big|_t^3\\ &= \lim_{t\to 1^+}\ln(8)-\ln|t^2-1|\\ &= \ln(8)-\lim_{x\to 1^+}\ln|t^2-1| = \infty.\end{aligned}$$

So this integral doesn't converge.

(b) Compute $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$.

Solution: We'll take $u=\sqrt{x}$ so $du=\frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}+x\sqrt{x}}=\frac{2du}{1+u^2}$. Then

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}} &= \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}+x\sqrt{x}}\\ &= \lim_{t\to\infty}\int_1^{\sqrt{t}}\frac{2du}{1+u^2}\\ &= \lim_{t\to\infty}2\arctan(u)\Big|_1^{\sqrt{t}} = \lim_{t\to\infty}2\arctan(\sqrt{t})-2\arctan(1)\\ &= 2(\pi/2)-2(\pi/4)=\pi/2.\end{aligned}$$

S5: Geometric Applications

(a) Set up (but don't evaluate!) an integral that gives the arc length of the curve $\arctan(x) = y^3$ as x varies from 1 to 6.

Solution: We have $y = \sqrt[3]{\arctan(x)}$ so $y' = \frac{1}{3}(\arctan(x))^{-2/3}\frac{1}{1+x^2}$, and thus

$$L = \int_1^6 \sqrt{1 + \frac{1}{9\arctan(x)^{4/3}(1+x^2)^2}}\,dx.$$

(b) Compute the area of the surface obtained by taking the curve $x^{2/3} + y^{2/3} = 1$ as x goes from 0 to 1 and rotating it around the y-axis.

Solution: We have $y = (1 - x^{2/3})^{3/2}$, so

$$\begin{aligned} y' &= \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot \frac{-2}{3}x^{-1/3} \\ &= \sqrt{1 - x^{2/3}}x^{-1/3} \\ (y')^2 &= (1 - x^{2/3})x^{-2/3} = x^{-2/3} - 1 \\ \sqrt{1 + (y')^2} &= \sqrt{x^{-2/3}} = x^{-1/3}. \end{aligned}$$

Then we compute

$$\begin{aligned} L &= \int_0^1 2\pi x\sqrt{1 + (y')^2}\,dx \\ &= 2\pi \int_0^1 x \cdot x^{-1/3}\,dx \\ &= 2\pi \int_0^1 x^{2/3}\,dx = 2\pi\frac{3}{5}x^{5/3}\Big|_0^1 \\ &= \frac{6\pi}{5}\left(1^{5/3} - 0^{5/3}\right) = \frac{6\pi}{5}. \end{aligned}$$

(If you graph the shape, you might notice that there's a symmetry, where we actually have $y = \pm(1 - x^{2/3})^{3/2}$. If you use both halves, you'll get twice this answer.)